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p -ADIC CONGRUENCES MOTIVATED BY SERIES

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ABSTRACT. Let $p > 5$ be a prime. Motivated by the known formulae

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3) \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{(2k+1)16^k} = \frac{4G}{\pi}$$

(where $G = \sum_{k=0}^{\infty} (-1)^k / (2k+1)^2$ is the Catalan constant), we show that

$$\begin{aligned} \sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} &\equiv -2B_{p-3} \pmod{p}, \\ \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{(2k+1)16^k} &\equiv -\frac{7}{4}p^2 B_{p-3} \pmod{p^3} \end{aligned}$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)16^k} \equiv -2q_p(2) - p q_p(2)^2 + \frac{5}{12}p^2 B_{p-3} \pmod{p^3},$$

where B_0, B_1, B_2, \dots are Bernoulli numbers and $q_p(2)$ is the Fermat quotient $(2^{p-1} - 1)/p$.

1. INTRODUCTION

Let $p > 3$ be a prime. In 2010 the author and R. Tauraso [ST] proved that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv \frac{8}{9}p^2 B_{p-3} \pmod{p^3},$$

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where B_0, B_1, B_2, \dots are Bernoulli numbers. Note that

$$\binom{2k}{k} = \frac{(2k)!}{k!^2} \equiv 0 \pmod{p} \quad \text{for } k = \frac{p+1}{2}, \dots, p-1.$$

The author [S11c] managed to show that

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3} E_{p-3} \pmod{p},$$

where E_0, E_1, E_2, \dots are Euler numbers. It is interesting to compare the last congruence with the known formula

$$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} = \frac{\zeta(2)}{3} = \frac{\pi^2}{18}.$$

In 1979 Apéry (cf. [Ap] and [Po]) proved the irrationality of $\zeta(3) = \sum_{k=1}^{\infty} 1/n^3$ and his following series plays an important role in the proof.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5} \zeta(3).$$

Motivated by this, Tauraso [T1] showed that if $p > 5$ is a prime then

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -\frac{2}{5} \cdot \frac{H_{p-1}}{p^2} \pmod{p^3}$$

and

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv \frac{4}{5} \cdot \frac{H_{p-1}}{p} \pmod{p^3},$$

where $H_n = \sum_{0 < k \leq n} 1/k$ ($n = 0, 1, 2, \dots$) are harmonic numbers. It is well known (cf. [G1] or [Su, Theorem 5.1(a)]) that

$$H_{p-1} \equiv -\frac{p^2}{3} B_{p-3} \pmod{p^3} \quad \text{for any prime } p > 3.$$

Actually Tauraso obtained $\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \pmod{p^4}$ for each prime $p > 5$ via putting $n = p$ in the following identity

$$\sum_{k=1}^n \binom{2k}{k} \frac{k^2}{4n^4 + k^4} \prod_{j=1}^{k-1} \frac{n^4 - j^4}{4n^4 + j^4} = \frac{2}{5n^2}$$

conjectured by J. M. Borwein and D. M. Bradley [BB] and proved by G. Almkvist and A. Granville [AG].

Let $p > 3$ be a prime. The author [S11c] proved that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}.$$

Recently Tauraso [T2] showed that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} \equiv -2H_{(p-1)/2} \pmod{p^3}.$$

Via *Mathematica* we find that

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2}{k16^k} = 4 \log 2 - \frac{8G}{\pi},$$

where G is the Catalan constant given by

$$G := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}.$$

Now we state our first theorem.

Theorem 1.1. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -2B_{p-3} \pmod{p}, \tag{1.1}$$

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv \frac{56}{15} p B_{p-3} \pmod{p^2}, \tag{1.2}$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k} \equiv -2H_{(p-1)/2} - \frac{7}{2} p^2 B_{p-3} \pmod{p^3}, \tag{1.3}$$

$$-\frac{4}{p^2} \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} \equiv \sum_{k=1}^{(p-1)/2} \frac{16^k}{k^3 \binom{2k}{k}^2} \equiv -14B_{p-3} \pmod{p}. \tag{1.4}$$

Remark 1.1. Both (1.1) and (1.2) were conjectured in [S11c, Conjecture 1.1]. The reader may consult [S10] and [S11a] for $\sum_{k=0}^{p-1} \binom{2k}{k}/m^k$ modulo powers of p , where p is an odd prime and m is an integer not divisible by p .

Motivated by the formulae

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} = \frac{\pi}{3} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} = \frac{\pi^2}{10},$$

the author [S11b] showed that

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv \frac{p}{3}E_{p-3} \pmod{p^2},$$

and conjectured that

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv \frac{H_{p-1}}{5p} \pmod{p^3}$$

(which was recently confirmed by Tauraso [T3]) and

$$\sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv -\frac{p}{4}B_{p-3} \pmod{p^2},$$

where p is any prime greater than 5.

Theorem 1.1 has the following consequence.

Corollary 1.1. *Let $p > 5$ be a prime. Then*

$$\frac{1}{p} \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv - \sum_{k=0}^{(p-3)/2} \frac{(-16)^k}{(2k+1)^3 \binom{2k}{k}} \equiv -\frac{B_{p-3}}{4} \pmod{p} \quad (1.5)$$

Since

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{(2k-1)16^k} = -\frac{2n+1}{16^n} \binom{2n}{n}^2 \quad \text{and} \quad \sum_{k=0}^n \frac{(1-4k)\binom{2k}{k}^4}{(2k-1)^4 256^k} = (8n^2+4n+1) \frac{\binom{2n}{n}^4}{256^n}$$

by induction, we find that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{(2k-1)16^k} = -\frac{2}{\pi} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{(4k-1)\binom{2k}{k}^4}{(2k-1)^4 256^k} = -\frac{8}{\pi^2}$$

by Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$. (The latter was first obtained by J. W. L. Glaisher [G2].) Via *Mathematica* (version 7) we find that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{(2k+1)16^k} = \frac{4G}{\pi} \quad \text{and} \quad \sum_{k=0}^{\infty} \frac{16^k}{(2k+1)^3 \binom{2k}{k}^2} = \frac{7}{2}\zeta(3) - G\pi.$$

Motivated by this we establish the following theorem.

Theorem 1.2. *Let $p > 3$ be a prime. Then*

$$\frac{1}{p^2} \sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{(2k+1)16^k} \equiv - \sum_{k=0}^{(p-3)/2} \frac{16^k}{(2k+1)^3 \binom{2k}{k}^2} \equiv -\frac{7}{4} B_{p-3} \pmod{p}, \quad (1.6)$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)16^k} \equiv -2q_p(2) - p q_p(2)^2 + \frac{5}{12} p^2 B_{p-3} \pmod{p^3}, \quad (1.7)$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$.

Now we pose two conjectures for further research.

Conjecture 1.1. *Let $p > 5$ be a prime. Then*

$$\sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} \equiv -\frac{21}{2} H_{p-1} \pmod{p^4}$$

and

$$\sum_{k=0}^{(p-3)/2} \frac{(-16)^k}{(2k+1)^3 \binom{2k}{k}} \equiv -\frac{3}{4} \cdot \frac{H_{p-1}}{p^2} - \frac{47}{400} p^2 B_{p-5} \pmod{p^3}.$$

Conjecture 1.2. *For any odd prime p , we have*

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{\binom{2k}{k} \binom{4k}{2k}}{k64^k} &\equiv -3H_{(p-1)/2} + \frac{7}{4} p^2 B_{p-3} \pmod{p^3}, \\ \sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k} \binom{4k}{2k}}{k64^k} &\equiv -3H_{(p-1)/2} + 2(-1)^{(p+1)/2} p E_{p-3} \pmod{p^2}, \\ p \sum_{k=1}^{(p-1)/2} \frac{64^k}{k^3 \binom{2k}{k} \binom{4k}{2k}} &\equiv (-1)^{(p-1)/2} 32 E_{p-3} \pmod{p}, \\ p \sum_{k=1}^{(p-1)/2} \frac{64^k}{(2k-1)k^2 \binom{2k}{k} \binom{4k}{k}} &\equiv 16 \left((-1)^{(p+1)/2} q_p(2) + p E_{p-3} \right) \pmod{p^2}. \end{aligned}$$

In the next section we are going to show Theorems 1.1 and Corollary 1.1. Theorem 1.2 will be proved in Section 3. Our proofs involve certain combinatorial identities and harmonic numbers of higher orders given by

$$H_n^{(m)} = \sum_{0 < k \leq n} \frac{1}{k^m} \quad (n = 0, 1, 2, \dots).$$

2. PROOF OF THEOREM 1.1

Lemma 2.1. *Let $p = 2n + 1$ be an odd prime. For any $k = 1, \dots, p - 1$, we have*

$$k \binom{2k}{k} \binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}. \quad (2.1)$$

and

$$\binom{n}{k} \binom{n+k}{k} \equiv \frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2} \quad (2.2)$$

Proof. (2.1) was formulated in [S11c, Lemma 2.1]; see also [T1] for such a trick. (2.2) is known and easy (see p. 231 of [vH, §3.4]); in fact,

$$\binom{n}{k} \binom{n+k}{k} (-1)^k = \binom{(p-1)/2}{k} \binom{(-p-1)/2}{k} \equiv \left(-\frac{1}{2}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.$$

We are done. \square

Lemma 2.2. *Let $p > 3$ be a prime. Then*

$$(-1)^{(p-1)/2} \binom{p-1}{(p-1)/2} \equiv 4^{p-1} + \frac{p^3}{12} B_{p-3} \pmod{p^4} \quad (2.3)$$

and

$$H_{(p-1)/2} \equiv -2q_p(2) + p q_p(2)^2 - p^2 \left(\frac{2}{3} q_p(2)^3 + \frac{7}{12} B_{p-3} \right) \pmod{p^3}. \quad (2.4)$$

Also,

$$H_{(p-1)/2}^{(2)} \equiv \frac{7}{3} p B_{p-3} \pmod{p^2} \quad \text{and} \quad H_{(p-1)/2}^{(3)} \equiv -2B_{p-3} \pmod{p}. \quad (2.5)$$

Remark 2.1. (2.3) and (2.4) are refinements of Morley's congruence and Lehmer's congruence given by L. Carlitz [C] and Z. H. Sun [Su, Theorem 5.2(c)] respectively. (2.5) follows from [Su, Corollary 5.2].

Lemma 2.3. *For $n = 1, 2, 3, \dots$ we have*

$$\sum_{k=1}^n \frac{(-1)^k}{k^3 \binom{n}{k} \binom{n+k}{k}} = 5 \sum_{k=1}^n \frac{(-1)^k}{k^3 \binom{2k}{k}} + 2H_n^{(3)} \quad (2.6)$$

and

$$\sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} \frac{(-1)^k}{k} (H_{n+k} - H_{n-k}) = \frac{5}{2} \sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{2k}{k} + 2H_n^{(2)}. \quad (2.7)$$

Proof. (2.6) is due to Apéry [Ap] (see also [Po]). The author found (2.7) via the math. software **Sigma**. (The reader may consult [OS, §5] for how to use **Sigma** to produce combinatorial identities.) In fact, if we let s_n denotes the left-hand side or the right-hand side of (2.7) then **Sigma** yields the recurrence relation

$$(n+1)^2(s_{n+1} - s_n) = 2 - 5(-1)^n \binom{2n+1}{n} \quad (n = 1, 2, 3, \dots).$$

So (2.7) can be proved by induction. \square

Lemma 2.4. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{k^2 16^k} \equiv -2H_{(p-1)/2}^2 \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{k^3 16^k} \equiv -\frac{4}{3}H_{(p-1)/2}^3 - \frac{2}{3}H_{(p-1)/2}^{(3)} \pmod{p}.$$

Remark 2.2. Lemma 2.4 is known, see [T2, Theorem 4.2] and its proof.

Lemma 2.5. *We have the new identity*

$$\sum_{k=0}^n \frac{\binom{2k}{k}^2}{(2(n+k)+1)16^k} = \frac{\binom{2n}{n}^2}{16^n} \sum_{k=0}^{2n} \frac{1}{2k+1}. \quad (2.8)$$

Proof. Let u_n denote the left-hand side or the right-hand side of (2.8). Applying the Zeilberger algorithm (cf. [PWZ]) via **Mathematica** (version 7), we find the recurrence relation

$$(2n+1)^2 u_n - 4(n+1)^2 u_{n+1} = -\frac{8(4n^3 + 8n^2 + 5n + 1)}{(4n+3)(4n+5)16^n} \binom{2n}{n}^2 \quad (n = 0, 1, 2, \dots).$$

So (2.7) holds by induction. \square

Proof of Theorem 1.1. For convenience we set $n = (p-1)/2$ and $S := \sum_{k=1}^n (-1)^k / (k^3 \binom{2k}{k})$. Below we divide the proof into two parts.

(i) In light of Lemma 2.1,

$$\frac{1}{p} \sum_{k=n+1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv \sum_{k=n+1}^{p-1} \frac{2(-1)^k}{k^3 \binom{2(p-k)}{p-k}} = \sum_{k=1}^n \frac{2(-1)^{p-k}}{(p-k)^3 \binom{2k}{k}} \equiv 2S \pmod{p}.$$

Hence (1.1) and (1.2) are equivalent since

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv -\frac{4}{15} p B_{p-3} \pmod{p^2} \quad (2.9)$$

by [T1].

By [S11c, (3.2)], for $k = 1, \dots, n$ we have

$$\binom{n}{k} \binom{n+k}{k} (-1)^k \left(1 - \frac{p}{4} (H_{n+k} - H_{n-k})\right) \equiv \frac{\binom{2k}{k}^2}{16^k} \pmod{p^4}.$$

This, together with (2.7) and the known identity

$$\sum_{k=1}^n \binom{n}{k} \binom{n+k}{k} \frac{(-1)^k}{k} = -2H_n$$

(cf. [Pr, §2.1]), yields that

$$\sum_{k=1}^n \frac{\binom{2k}{k}^2}{k 16^k} + 2H_n \equiv -\frac{p}{4} \left(\frac{5}{2} \sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{2k}{k} + 2H_n^{(2)} \right) \pmod{p^4}.$$

Thus, in view of the first congruence in (2.5), we have

$$\sum_{k=1}^n \frac{\binom{2k}{k}^2}{k 16^k} + 2H_n \equiv -\frac{5}{8} p \sum_{k=1}^n \frac{(-1)^k}{k^2} \binom{2k}{k} - \frac{7}{6} p^2 B_{p-3} \pmod{p^3}. \quad (2.10)$$

Therefore (1.2) and (1.3) are equivalent.

In light of Lemma 2.1,

$$\begin{aligned} \frac{1}{p^2} \sum_{k=n+1}^{p-1} \frac{\binom{2k}{k}^2}{k 16^k} &\equiv \sum_{k=n+1}^{p-1} \frac{4}{k^3 16^k \binom{2(p-k)}{p-k}^2} = \sum_{k=1}^n \frac{4}{(p-k)^3 16^{p-k} \binom{2k}{k}^2} \\ &\equiv -\frac{1}{4} \sum_{k=1}^n \frac{16^k}{k^3 \binom{2k}{k}^2} \pmod{p}. \end{aligned}$$

With the helps of (2.2) and (2.5), we obtain from (2.6) the congruence

$$\sum_{k=1}^n \frac{16^k}{k^3 \binom{2k}{k}^2} \equiv 5S - 4B_{p-3} \pmod{p}.$$

Therefore (1.1) and (1.4) are equivalent.

Note that

$$\sum_{k=n+1}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} \equiv -\frac{p^2}{4}(5S-4B_{p-3}) \equiv -\frac{5}{8}p \sum_{k=n+1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} + p^2 B_{p-3} \pmod{p^3}.$$

Combining this with (2.10) we get

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k16^k} + 2H_n \equiv -\frac{5}{8}p \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} - \frac{p^2}{6} B_{p-3} \pmod{p^3}. \quad (2.11)$$

So Tauraso's result $\sum_{k=1}^{p-1} \binom{2k}{k}^2 / (k16^k) \equiv -2H_n \pmod{p^3}$ in [T2] actually follows from his earlier result (2.9).

(ii) By the above, it suffices to show (1.3). Note that

$$\sum_{k=0}^{2n} \frac{1}{2k+1} = H_{4n+1} - \frac{H_{2n}}{2} = H_{p-1} + \frac{1}{p} + \sum_{k=1}^{p-1} \frac{1}{p+k} - \frac{H_{p-1}}{2}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{p+k} = \sum_{k=1}^{p-1} \frac{k^2 - kp + p^2}{k^3 + p^3} \equiv \sum_{k=1}^{p-1} \frac{k^2 - kp + p^2}{k^3} = H_{p-1} - pH_{p-1}^{(2)} + p^2 H_{p-1}^{(3)}.$$

As

$$H_{p-1}^{(3)} = \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k^3} + \frac{1}{(p-k)^3} \right) \equiv 0 \pmod{p},$$

we see that

$$\sum_{k=0}^{2n} \frac{1}{2k+1} - \frac{1}{p} \equiv \frac{3}{2}H_{p-1} - pH_{p-1}^{(2)} \equiv -\frac{p^2}{2}B_{p-3} - \frac{2}{3}p^2B_{p-3} = -\frac{7}{6}p^2B_{p-3} \pmod{p^3}$$

since $H_{p-1} \equiv -p^2B_{p-3}/3 \pmod{p^3}$ and $H_{p-1}^{(2)} \equiv 2pB_{p-3}/3 \pmod{p^2}$ by Glaisher [G1] (see also [Su, Theorem 5.1(a)]). In view of (2.3),

$$\begin{aligned} \frac{\binom{2n}{n}^2}{16^n} &\equiv \frac{(4^{p-1}(1 + p^3B_{p-3}/12))^2}{4^{p-1}} \equiv 4^{p-1} \left(1 + \frac{p^3}{6}B_{p-3} \right) \\ &\equiv (1 + pq_p(2))^2 + \frac{p^3}{6}B_{p-3} \pmod{p^4}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{\binom{2n}{n}^2}{16^n} \sum_{k=0}^{2n} \frac{1}{2k+1} - \frac{1}{p} &= \frac{\binom{2n}{n}^2}{16^n} \left(\sum_{k=0}^{2n} \frac{1}{2k+1} - \frac{1}{p} \right) + \frac{\binom{2n}{n}^2/16^n - 1}{p} \\ &\equiv -\frac{7}{6}p^2 B_{p-3} + \frac{(1 + p q_p(2))^2 + p^3 B_{p-3}/6 - 1}{p} \\ &= 2q_p(2) + p q_p(2)^2 - p^2 B_{p-3} \pmod{p^3} \end{aligned}$$

and hence (2.8) yields the congruence

$$\sum_{k=1}^n \frac{\binom{2k}{k}^2}{(2k+p)16^k} \equiv 2q_p(2) + p q_p(2)^2 - p^2 B_{p-3} \pmod{p^3}. \quad (2.12)$$

Note that

$$\begin{aligned} \sum_{k=1}^n \frac{\binom{2k}{k}^2}{(2k+p)16^k} &= \sum_{k=1}^n \frac{(4k^2 - 2kp + p^2)\binom{2k}{k}^2}{((2k)^3 + p^3)16^k} \\ &\equiv \frac{1}{2} \sum_{k=1}^n \frac{\binom{2k}{k}^2}{k16^k} - \frac{p}{4} \sum_{k=1}^n \frac{\binom{2k}{k}^2}{k^2 16^k} + \frac{p^2}{8} \sum_{k=1}^n \frac{\binom{2k}{k}^2}{k^3 16^k} \pmod{p^3}. \end{aligned} \quad (2.13)$$

By Lemma 2.4 and (2.4)-(2.5),

$$\sum_{k=1}^n \frac{\binom{2k}{k}^2}{k^2 16^k} \equiv -2(-2q_p(2) + p q_p(2)^2)^2 \equiv -8q_p(2)^2 + 8p q_p(2)^3 \pmod{p^2} \quad (2.14)$$

and

$$\sum_{k=1}^n \frac{\binom{2k}{k}^2}{k^3 16^k} \equiv -\frac{4}{3}(-2q_p(2))^3 - \frac{2}{3}(-2B_{p-3}) = \frac{32}{3}q_p(2)^3 + \frac{4}{3}B_{p-3} \pmod{p}. \quad (2.15)$$

Combining (2.12)-(2.15) we obtain

$$\begin{aligned} \frac{1}{2} \sum_{k=1}^n \frac{\binom{2k}{k}^2}{k16^k} &\equiv 2q_p(2) + p q_p(2)^2 - p^2 B_{p-3} + \frac{p}{4}(-8q_p(2)^2 + 8p q_p(2)^3) \\ &\quad - \frac{p^2}{8} \left(\frac{32}{3}q_p(2)^3 + \frac{4}{3}B_{p-3} \right) \pmod{p^3}. \end{aligned}$$

In view of (2.4), this yields the desired (1.3).

The proof of Theorem 1.1 is now complete. \square

Proof of Corollary 1.1. Write $p = 2n + 1$. In view of Lemma 2.1,

$$\begin{aligned} & \frac{1}{p} \sum_{k=n+1}^{p-1} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \\ &= \sum_{k=1}^n \frac{\binom{2(p-k)}{p-k}/p}{(2(p-k)+1)^2(-16)^{p-k}} \\ &\equiv \sum_{k=1}^n \frac{-2/(k\binom{2k}{k})}{(2(k-1)+1)^2(-16)^{1-k}} = - \sum_{k=0}^{n-1} \frac{(-16)^k}{(2k+1)^3\binom{2k}{k}} \pmod{p}. \end{aligned}$$

Since

$$\binom{-1/2}{k} \equiv \binom{n}{k} = \binom{n}{n-k} \equiv \binom{-1/2}{n-k} \pmod{p} \quad \text{for all } k = 0, \dots, n,$$

we have

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{(-16)^k}{(2k+1)^3\binom{2k}{k}} &= \sum_{k=0}^{n-1} \frac{4^k}{(2k+1)^3\binom{-1/2}{k}} \\ &\equiv \sum_{k=0}^{n-1} \frac{4^k}{(2k+1)^3\binom{-1/2}{n-k}} = \sum_{k=1}^n \frac{4^{n-k}}{(2(n-k)+1)^3\binom{-1/2}{k}} \\ &\equiv -\frac{1}{8} \sum_{k=1}^n \frac{1}{k^3\binom{-1/2}{k}4^k} = -\frac{1}{8} \sum_{k=1}^n \frac{(-1)^k}{k^3\binom{2k}{k}} \\ &\equiv \frac{B_{p-3}}{4} \pmod{p} \quad (\text{by (1.1)}). \end{aligned}$$

Therefore (1.5) holds. \square

3. PROOF OF THEOREM 1.2

Lemma 3.1. *For any positive integer n we have*

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(n-k)16^k} = \frac{\binom{2n}{n}^2}{4^{2n-1}} \sum_{k=0}^{n-1} \frac{1}{2k+1} \quad (3.1)$$

and

$$\sum_{k=0}^n \frac{(-1)^k}{(2k+1)^2} \binom{n}{k} \binom{n+k}{k} = \frac{1}{(2n+1)^2} + \frac{2}{2n+1} \sum_{k=0}^{n-1} \frac{1}{2k+1}. \quad (3.2)$$

Proof. Let $(x)_0 = 1$ and $(x)_k = \prod_{j=0}^{k-1} (x+j)$ for $k = 1, 2, 3, \dots$. Then $(1/2)_k^2 / (1)_k^2 = \binom{2k}{k}^2 / 16^k$ for $k = 0, 1, 2, \dots$. Thus (3.1) is just (21) of [Lu, Ch. 5.2] with $x = 1/2$ (see also [T2, Theorem 2.1]).

(3.2) is new and it can be proved via the Zeilberger algorithm (cf. [PWZ]). It is easy to verify (3.2) for $n = 1, 2$. Applying the Zeilberger algorithm via **Mathematica** (version 7), we find that if a_n denotes the left-hand side or the right-hand side of (3.2) then

$$(n+1)(2n+5)^2 a_{n+2} = (2n+3)(4n^2+12n+7)a_{n+1} - (n+2)(2n+1)^2 a_n$$

for all $n = 1, 2, 3, \dots$. So (3.2) follows by induction. \square

Lemma 3.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)^3 16^k} \equiv -\frac{4}{3} q_p(2)^3 - \frac{B_{p-3}}{6} \pmod{p}. \quad (3.3)$$

Proof. Set $n = (p-1)/2$. For $k = 0, \dots, n$ we clearly have

$$\binom{n+k}{k} (-1)^k = \binom{-n-1}{k} \equiv \binom{n}{k} \equiv \binom{-1/2}{k} = \frac{\binom{2k}{k}}{(-4)^k} \pmod{p}.$$

Thus

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(2k+1)^3 16^k} &\equiv \sum_{k=0}^{n-1} \frac{\binom{n}{k}^2}{(2k+1)^3} = \sum_{k=1}^n \frac{\binom{n}{k}^2}{(2(n-k)+1)^3} \\ &\equiv -\frac{1}{8} \sum_{k=1}^n \frac{\binom{2k}{k}^2}{k^3 16^k} \pmod{p}. \end{aligned}$$

Combining this with (2.15) we obtain the desired (3.3). \square

Lemma 3.3. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)^2 16^k} \equiv -2q_p(2)^2 + \frac{2}{3} p q_p(2)^3 - \frac{p}{6} B_{p-3} \pmod{p^2}. \quad (3.4)$$

Proof. Write $p = 2n + 1$. By (3.2) we have

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{(-1)^k}{(2k+1)^2} \binom{n}{k} \binom{n+k}{k} &= \frac{1 - (-1)^n \binom{2n}{n}}{p^2} + \frac{2}{p} \left(H_{p-1} - \frac{H_n}{2} \right) \\ &= \frac{1 - (-1)^n \binom{2n}{n} - p H_n}{p^2} + \frac{2}{p} H_{p-1} \end{aligned}$$

In light of Lemma 2.2,

$$\begin{aligned} 1 - (-1)^n \binom{2n}{n} - pH_n &\equiv 1 - (1 + pq_p(2))^2 - \frac{p^3}{12} B_{p-3} \\ &\quad + 2pq_p(2) - p^2 q_p(2)^2 + p^3 \left(\frac{2}{3} q_p(2)^3 + \frac{7}{12} B_{p-3} \right) \\ &\equiv -2p^2 q_p(2)^2 + p^3 \left(\frac{2}{3} q_p(2)^3 + \frac{B_{p-3}}{2} \right) \pmod{p^4}. \end{aligned}$$

So, with the help of (2.2), we get

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(2k+1)^2 16^k} \equiv -2q_p(2)^2 + p \left(\frac{2}{3} q_p(2)^3 + \frac{B_{p-3}}{2} \right) + \frac{2}{p} H_{p-1} \pmod{p^2},$$

which gives (3.4) since $H_{p-1}/p \equiv -pB_{p-3}/3 \pmod{p^2}$. \square

Proof of Theorem 1.2. For convenience we set $n = (p-1)/2$.

We first prove (1.6). By Lemma 2.1,

$$\begin{aligned} \frac{1}{p^2} \sum_{k=n+1}^{p-1} \frac{\binom{2k}{k}^2}{(2k+1)16^k} &= \sum_{k=1}^n \frac{(\binom{2(p-k)}{p-k}/p)^2}{(2(p-k)+1)16^{p-k}} \\ &\equiv \sum_{k=1}^n \frac{(-2/(k\binom{2k}{k}))^2 16^{k-1}}{1-2k} = - \sum_{k=0}^{n-1} \frac{16^k}{(2k+1)^3 \binom{2k}{k}^2} \pmod{p}. \end{aligned}$$

Note also that

$$\begin{aligned} \sum_{k=0}^{n-1} \frac{16^k}{(2k+1)^3 \binom{2k}{k}^2} &\equiv \sum_{k=0}^{n-1} \frac{1}{(2k+1)^3 \binom{n}{k}^2} = \sum_{k=1}^n \frac{1}{(2(n-k)+1)^3 \binom{n}{k}^2} \\ &\equiv -\frac{1}{8} \sum_{k=1}^n \frac{1}{k^3 \binom{n}{k}^2} \equiv -\frac{1}{8} \sum_{k=1}^n \frac{16^k}{k^3 \binom{2k}{k}^2} \pmod{p}. \end{aligned}$$

Thus, with the help (1.4) we get (1.6). (In the case $p = 5$, (1.6) can be verified directly.)

Since $\binom{2n}{n}^2 \equiv 4^{4n} \pmod{p^3}$ by (2.3), we have

$$\begin{aligned} \frac{\binom{2n}{n}^2}{4^{2n-1}} \sum_{k=0}^{n-1} \frac{1}{2k+1} &\equiv 4^p \left(H_{p-1} - \frac{H_n}{2} \right) = 4^p H_{p-1} - 2(1 + pq_p(2))^2 H_n \\ &\equiv -\frac{4}{3} p^2 B_{p-3} - 2(1 + 2pq_p(2) + p^2 q_p(2)^2) H_n \\ &\equiv 4q_p(2) + 6pq_p(2)^2 + p^2 \left(\frac{4}{3} q_p(2)^3 - \frac{B_{p-3}}{6} \right) \pmod{p^3} \end{aligned}$$

with the help of (2.4). Note also that

$$\begin{aligned} & \frac{1}{2} \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(n-k)16^k} = \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(p-(2k+1))16^k} \\ &= \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2 (p^2 + p(2k+1) + (2k+1)^2)}{(p^3 - (2k+1)^3)16^k} \\ &\equiv -p^2 \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(2k+1)^3 16^k} - p \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(2k+1)^2 16^k} - \sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(2k+1)16^k} \pmod{p^3}. \end{aligned}$$

Combining these with (3.1), (3.3) and (3.4) we finally obtain the desired (1.7). \square

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