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SOME NEW SUPER-CONGRUENCES MODULO PRIME POWERS

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Abstract. Let $p>3$ be a prime. We show that

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1}\equiv 0 \pmod{p^5} \ \text{and}\ \sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1}\equiv 0 \pmod{p^4}.$$

For any positive integer $m\not\equiv 0 \pmod p$, we prove that

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m\equiv 0 \pmod{p^4},$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^2}\binom{p/m-1}{k}^m\equiv \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{k} \pmod{p^3} \quad \text{if } p>5.$$

The paper also contains some open conjectures.

1. Introduction

A p-adic congruence (with p a prime) is called a *super-congruence* if it happens to hold modulo higher powers of p. Here is a classical example due to J. Wolstenholme (cf. [W] or [HT]):

$$\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2} \ \text{and}\ \binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$$

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for any prime $p > 3$. The reader may consult Long [L], Sun [Su1] and [Su2] for some known super-congruences.

In this paper we obtain some new super-congruences modulo prime powers motivated by the well-known formula

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Now we state our main results.

Theorem 1.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv 0 \pmod{p^5}. \tag{1.1}$$

Theorem 1.2. *Let $p > 3$ be a prime and let m be a positive integer not divisible by p. Then we have*

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m \equiv 0 \pmod{p^4}. \tag{1.2}$$

In particular,

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv 0 \pmod{p^4}. \tag{1.3}$$

Remark 1.1. We conjecture that there are no composite numbers p satisfying (1.1) or (1.3). We also note that (1.1) and (1.3) can be refined as follows:

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^5}{18}B_{p-3} \pmod{p^6} \quad \text{for any prime } p > 5, \tag{1.4}$$

and

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv \frac{2}{3}p^4 B_{p-3} \pmod{p^5} \quad \text{for any prime } p > 3, \tag{1.5}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers (see [IR, pp. 228–241] for an introduction to Bernoulli numbers). However, the proofs of (1.4) and (1.5) are too complicated.

Theorem 1.3. *Let $p>3$ be a prime and let m be a positive integer not divisible by p.*

(i) *If $p>5$, then*

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^2}\binom{p/m-1}{k}^m\equiv\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{k}\pmod{p^3}. \tag{1.6}$$

Also, for any $n=1,\ldots,(p-3)/2$ we have

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n}}\binom{p/m-1}{k}^m\equiv-\frac{p}{2n+1}B_{p-1-2n}\pmod{p^2}. \tag{1.7}$$

(ii) *For every $n=1,\ldots,(p-3)/2$, we have*

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n-1}}\binom{p/m-1}{k}^m\equiv\left(1+\frac{1-m}{2m}(2n+1)\right)\frac{p^2n}{2n+1}B_{p-1-2n}\pmod{p^3}. \tag{1.8}$$

Remark 1.2. If n is a positive integer and $p>2n+1$ is a prime then (1.8) with $m=p\pm1$ yields

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\binom{1/(p-1)}{k}^{p-1}\equiv-\frac{2p^2n^2}{2n+1}B_{p-1-2n}\pmod{p^3} \tag{1.9}$$

and

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\binom{-1/(p+1)}{k}^{p+1}\equiv\frac{p^2n}{2n+1}B_{p-1-2n}\pmod{p^3}. \tag{1.10}$$

For a prime p and a p-adic number x, as usual we let $\nu_p(x)$ denote the p-adic valuation (i.e., p-adic order) of x.

Conjecture 1.1. *Let p be a prime and let n be a positive integer. Then*

$$\nu_p\Bigg(\sum_{k=0}^{n-1}\binom{-1/(p+1)}{k}^{p+1}\Bigg)\geqslant c_p\left\lfloor\frac{\nu_p(n)+1}{2}\right\rfloor,$$

where

$$c_p=\begin{cases}1 & \text{if } p=2,\\ 3 & \text{if } p=3,\\ 5 & \text{if } p\geqslant 5.\end{cases}$$

If $p>3$ then

$$\nu_p\Bigg(\sum_{k=0}^{n-1}\binom{1/(p-1)}{k}^{p-1}\Bigg)\geqslant 4\left\lfloor\frac{\nu_p(n)+1}{2}\right\rfloor.$$

Now we raise one more conjecture.

Conjecture 1.2. *Let $m \geqslant 2$ and r be integers. And let $p > r$ be an odd prime not dividing m.*

(i) *If $m > 2$, $m \not\equiv r \pmod 2$, and $p \equiv r \pmod m$ with $r \geqslant -m/2$, then*

$$\sum_{k=0}^{p-1}(-1)^{km}\binom{r/m}{k}^m \equiv 0 \pmod{p^3}. \tag{1.11}$$

(ii) *If $p \equiv r \pmod{2m}$ with $r \geqslant -m$, then*

$$\sum_{k=0}^{p-1}(-1)^k\binom{r/m}{k}^{2n+1} \equiv 0 \pmod{p^2} \quad \text{for all } n = 1, \ldots, m-1. \tag{1.12}$$

Remark 1.3. The congruences in (1.11) and (1.12) modulo p are easy.

Theorems 1.1-1.2 and Theorem 1.3 will be proved in Sections 2 and 3 respectively.

2. Proofs of Theorems 1.1 and 1.2

For $m = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$, we define

$$H_n^{(m)} := \sum_{0<k\leqslant n}\frac{1}{k^m}$$

and call it a harmonic number of order m. Those $H_n = H_n^{(1)}$ $(n = 0, 1, 2, \ldots)$ are usually called *harmonic numbers.*

Lemma 2.1. *Let $p > 3$ be a prime. Then*

$$H_{p-1} \equiv -\frac{p^2}{3}B_{p-3} \pmod{p^3}, \quad H_{p-1}^{(2)} \equiv \frac{2}{3}pB_{p-3} \pmod{p^2}, \tag{2.1}$$

and

$$\sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant p-1}\frac{1}{i_1 i_2\cdots i_k} \equiv \frac{(-1)^{k-1}}{k+1}pB_{p-1-k} \pmod{p^2} \tag{2.2}$$

for all $k = 1, 2, \ldots, p-1$.

Remark 2.1. The results in Lemma 2.1 are known, see Theorem 5.1, Corollary 5.1 and the proof of [S, Theorem 6.1].

Lemma 2.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1} H_k \equiv -\frac{p^3}{3}B_{p-3} - p + 1 \pmod{p^4}, \tag{2.3}$$

and

$$\sum_{k=1}^{p-1} H_k^{(2)} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} H_k^{(3)} \equiv 0 \pmod{p}. \tag{2.4}$$

Proof. For any positive integer m we have

$$\sum_{k=1}^{p-1} H_k^{(m)} = \sum_{k=1}^{p-1}\sum_{j=1}^{k}\frac{1}{j^m} = \sum_{j=1}^{p-1}\frac{\sum_{k=j}^{p-1}1}{j^m} = \sum_{j=1}^{p-1}\frac{p-j}{j^m}.$$

Thus

$$\sum_{k=1}^{p-1} H_k = pH_{p-1}-p+1,\ \sum_{k=1}^{p-1} H_k^{(2)} = pH_{p-1}^{(2)}-H_{p-1},\ \sum_{k=1}^{p-1} H_k^{(3)} = pH_{p-1}^{(3)}-H_{p-1}^{(2)}.$$

Combining these with (2.1), we immediately get (2.3) and (2.4). $\square$

Lemma 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij} \equiv -\frac{2}{3}p^2 B_{p-3} + p - 1 \pmod{p^3}, \tag{2.5}$$

$$\sum_{k=1}^{p-1}\sum_{1\leqslant i_1<i_2<i_3\leqslant k}\frac{1}{i_1 i_2 i_3} \equiv -\frac{p}{3}B_{p-3} - p + 1 \pmod{p^2}, \tag{2.6}$$

and

$$\sum_{k=1}^{p-1}\sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant k}\frac{1}{i_1 i_2 i_3 i_4} \equiv -1 \pmod{p}. \tag{2.7}$$

We also have

$$\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2 j}\right) \equiv 0 \pmod{p}. \tag{2.8}$$

Proof. For $s = 2, \ldots, p-1$ it is clear that

$$\begin{aligned}
&\sum_{k=1}^{p-1} \sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant k} \frac{1}{i_1 i_2 \cdots i_s} \\
=&\sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1} \frac{\sum_{k=i_s}^{p-1} 1}{i_1 i_2 \cdots i_s} = \sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1} \frac{p-i_s}{i_1 i_2 \cdots i_s} \\
=&p \sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1} \frac{1}{i_1 i_2 \cdots i_s} - \sum_{1\leqslant i_1<i_2<\cdots<i_{s-1}\leqslant p-1} \frac{\sum_{i_{s-1}<i_s<p} 1}{i_1 i_2 \cdots i_{s-1}} \\
=&p \sum_{1\leqslant i_1<i_2<\cdots<i_s\leqslant p-1} \frac{1}{i_1 i_2 \cdots i_s} - \sum_{1\leqslant i_1<i_2<\cdots<i_{s-1}\leqslant p-1} \frac{p-1-i_{s-1}}{i_1 i_2 \cdots i_{s-1}}.
\end{aligned}$$

Thus, with the help of Lemmas 2.1, we have

$$\begin{aligned}
\sum_{k=1}^{p-1} \sum_{1\leqslant i<j\leqslant k} \frac{1}{ij} =&p \sum_{1\leqslant i<j\leqslant p-1} \frac{1}{ij} - \sum_{i=1}^{p-1} \frac{p-1-i}{i} \\
\equiv&-\frac{p^2}{3} B_{p-3} - (p-1)H_{p-1} + p - 1 \\
\equiv&-\frac{2}{3} p^2 B_{p-3} + p - 1 \pmod{p^3}.
\end{aligned}$$

Also,

$$\begin{aligned}
&\sum_{k=1}^{p-1} \sum_{1\leqslant i_1<i_2<i_3\leqslant k} \frac{1}{i_1 i_2 i_3} \\
=&p \sum_{1\leqslant i_1<i_2<i_3\leqslant p-1} \frac{1}{i_1 i_2 i_3} - \sum_{1\leqslant i_1<i_2\leqslant p-1} \frac{p-1-i_2}{i_1 i_2} \\
\equiv&-(p-1) \sum_{1\leqslant i_1<i_2\leqslant p-1} \frac{1}{i_1 i_2} + \sum_{i_1=1}^{p-1} \frac{\sum_{i_1<i_2<p} 1}{i_1} \\
\equiv&(p-1)\frac{p}{3} B_{p-3} + \sum_{i=1}^{p-1} \frac{p-1-i}{i} \equiv -\frac{p}{3} B_{p-3} - p + 1 \pmod{p^2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{p-1}\sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant k}\frac{1}{i_1i_2i_3i_4}\\
=&p\sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant p-1}\frac{1}{i_1i_2i_3i_4}-\sum_{1\leqslant i_1<i_2<i_3\leqslant p-1}\frac{p-1-i_3}{i_1i_2i_3}\\
\equiv&\sum_{1\leqslant i_1<i_2<i_3\leqslant p-1}\frac{1+i_3}{i_1i_2i_3}\equiv\sum_{1\leqslant i_1<i_2\leqslant p-1}\frac{\sum_{i_2<i_3<p}1}{i_1i_2}\\
\equiv&\sum_{1\leqslant i<j\leqslant p-1}\frac{p-1-j}{ij}\equiv-\sum_{i=1}^{p-1}\frac{\sum_{i<j<p}1}{i}=-\sum_{i=1}^{p-1}\frac{p-1-i}{i}\equiv-1 \pmod p.
\end{aligned}$$

Finally, we note that

$$\begin{aligned}
&\sum_{k=1}^{p-1}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right)=\sum_{1\leqslant i<j\leqslant p-1}\left(\frac{p-j}{ij^2}+\frac{p-j}{i^2j}\right)\\
\equiv&-\sum_{1\leqslant i<j\leqslant p-1}\left(\frac{1}{ij}+\frac{1}{i^2}\right)\equiv-\sum_{i=1}^{p-1}\frac{\sum_{i<j<p}1}{i^2}\\
=&-\sum_{i=1}^{p-1}\frac{p-1-i}{i^2}=H_{p-1}-(p-1)H_{p-1}^{(2)}\equiv 0 \pmod p
\end{aligned}$$

with the help of (2.1).

So far we have proved (2.5)-(2.8). □

Proof of Theorem 1.1. For each $k=1,\ldots,p-1$, clearly

$$\begin{aligned}
&\binom{-1/(p+1)}{k}^{p+1}=\binom{p/(p+1)-1}{k}^{p+1}=\prod_{j=1}^{k}\left(1-\frac{p}{(p+1)j}\right)^{p+1}\\
\equiv&\prod_{j=1}^{k}\left(1-\frac{(p+1)p}{(p+1)j}+\frac{(p+1)p}{2}\cdot\frac{p^2}{(p+1)^2j^2}-\frac{(p+1)p(p-1)}{3!}\cdot\frac{p^3}{(p+1)^3j^3}\right)\\
=&\prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{p^3}{2(p+1)j^2}-\frac{p^4(p-1)}{6(p+1)^2j^2}\right)\\
\equiv&\prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{p^3(1-p)}{2j^2}+\frac{p^4}{6j^3}\right) \pmod{p^5}
\end{aligned}$$

and hence

$$\begin{aligned}\binom{-1/(p+1)}{k}^{p+1} \equiv& 1 - pH_k + \frac{p^3(1-p)}{2}H_k^{(2)} + \frac{p^4}{6}H_k^{(3)} + p^2 \sum_{1\leqslant i<j\leqslant k}\frac{1}{ij} \\ &- p^3 \sum_{1\leqslant i_1<i_2<i_3\leqslant k}\frac{1}{i_1i_2i_3} + p^4 \sum_{1\leqslant i_1<i_2<i_3<i_4\leqslant k}\frac{1}{i_1i_2i_3i_4} \\ &- \frac{p^4}{2}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right) \pmod{p^5}.\end{aligned}$$

Thus, in view of Lemmas 2.1-2.3, we obtain

$$\begin{aligned}&\sum_{k=1}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \\ \equiv& p-1-p\left(-\frac{p^3}{3}B_{p-3}-p+1\right)+p^2\left(-\frac{p}{3}B_{p-3}-p+1\right) \\ &-p^3\left(-\frac{p}{3}B_{p-3}-p+1\right)-p^4 \\ \equiv& -1 \pmod{p^5}\end{aligned}$$

and hence (1.1) follows.

The proof of Theorem 1.1 is now complete. □

Proof of Theorem 1.2. For each $k \in \{1, \ldots, p-1\}$, we have

$$\begin{aligned}(-1)^{km}\binom{p/m-1}{k}^m =& \prod_{j=1}^k\left(1-\frac{p}{jm}\right)^m \\ \equiv& \prod_{j=1}^k\left(1-\frac{pm}{jm}+\frac{m(m-1)}{2}\cdot\frac{p^2}{j^2m^2}-\binom{m}{3}\frac{p^3}{j^3m^3}\right) \\ \equiv& \prod_{j=1}^k\left(1-\frac{p}{j}+\frac{m-1}{2m}\cdot\frac{p^2}{j^2}-\frac{(m-1)(m-2)}{6m^2}\cdot\frac{p^3}{j^3}\right) \\ \equiv& 1-pH_k+\frac{m-1}{2m}p^2H_k^{(2)}-\frac{(m-1)(m-2)}{6m^2}p^3H_k^{(3)} \\ &+p^2\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij}-p^3\sum_{1\leqslant i_1<i_2<i_3\leqslant k}\frac{1}{i_1i_2i_3} \\ &-\frac{m-1}{2m}p^3\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right) \pmod{p^4}.\end{aligned}$$

Therefore, applying Lemmas 2.2 and 2.3 we get

$$\sum_{k=1}^{p-1}(-1)^{km}\binom{p/m-1}{k}^m \equiv p-1-p(-p+1)+p^2(p-1)-p^3 = -1 \pmod{p^4}.$$

This proves (1.2). Clearly (1.2) in the case $m = p-1$ yields (1.3). This concludes the proof. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let m and n be positive integers with $m \leqslant 2n+1$, and let $p > 2n+1$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{H_k^{(m)}}{k^{2n+1-m}} \equiv \frac{(-1)^{m-1}}{2n+1}\binom{2n+1}{m} B_{p-1-2n} \pmod{p}. \tag{3.1}$$

When $m < 2n$ we have

$$\sum_{k=1}^{p-1} \frac{H_k^{(m)}}{k^{2n-m}} \equiv \frac{pB_{p-1-2n}}{2n+1}\left(n + (-1)^m \frac{n-m}{m+1}\binom{2n+1}{m}\right) \pmod{p^2}. \tag{3.2}$$

Proof. Since $\sum_{k=1}^{p-1} k^s \equiv 0 \pmod{p}$ for any integer $s \not\equiv 0 \pmod{p-1}$ (see, e.g., [IR, p. 235]), we have $\sum_{k=1}^{p-1} 1/k^{2n+1} \equiv 0 \pmod{p}$. Hence

$$\begin{aligned}
&\sum_{k=1}^{p-1} \frac{H_k^{(m)}}{k^{2n+1-m}} = \sum_{k=1}^{p-1}\frac{1}{k^{2n+1}} + \sum_{k=1}^{p-1}\frac{1}{k^{2n+1-m}}\sum_{j=0}^{p-1} j^{p-1-m} \\
\equiv &\sum_{k=1}^{p-1} \frac{1}{k^{2n+1-m}(p-m)} \sum_{i=0}^{p-1-m}\binom{p-m}{i} B_i k^{p-m-i} \\
\equiv &-\frac{1}{m}\sum_{i=0}^{p-1-m}\binom{p-m}{i} B_i \sum_{k=1}^{p-1} k^{p-1-2n-i} \\
\equiv &\frac{1}{m}\sum_{\substack{0\leqslant i\leqslant p-1-m\\ p-1\mid 2n+i}}\binom{p-m}{i} B_i = \frac{1}{m}\binom{p-m}{2n+1-m} B_{p-1-2n} \\
\equiv &\frac{1}{m}\binom{-m}{2n+1-m} B_{p-1-2n} = \frac{(-1)^{m-1}}{2n+1}\binom{2n+1}{m} B_{p-1-2n} \pmod{p}.
\end{aligned}$$

This proves (3.1).

Now assume that $m < 2n$. As $m, 2n-m \in \{1, \ldots, p-2\}$, we have

$$\sum_{j=1}^{p-1} \frac{1}{j^m} \equiv \sum_{k=1}^{p-1} \frac{1}{k^{2n-m}} \equiv 0 \pmod{p}.$$

It is known that

$$\sum_{k=0}^{p-1} \frac{1}{k^s} \equiv \frac{ps}{s+1} B_{p-1-s} \pmod{p^2} \quad \text{for each } s = 1, \ldots, p-2. \tag{3.3}$$

(See, e.g., [S, Corollary 5.1].) Thus

$$\sum_{j=1}^{p-1}\frac{1}{j^m}\sum_{k=1}^{p-1}\frac{1}{k^{2n-m}}+\sum_{k=1}^{p-1}\frac{1}{k^{2n}}\equiv\sum_{k=1}^{p-1}\frac{1}{k^{2n}}\equiv\frac{2n}{2n+1}pB_{p-1-2n}\pmod{p^2}.$$

On the other hand,

$$\begin{aligned}&\sum_{j=1}^{p-1}\frac{1}{j^m}\sum_{k=1}^{p-1}\frac{1}{k^{2n-m}}+\sum_{k=1}^{p-1}\frac{1}{k^{2n}}-\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{j^mk^{2n-m}}\\ =&\sum_{1\leqslant k\leqslant j\leqslant p-1}\frac{1}{j^mk^{2n-m}}=\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{1}{(p-j)^m(p-k)^{2n-m}}\\ =&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{(p+j)^m(p+k)^{2n-m}}{(p^2-j^2)^m(p^2-k^2)^{2n-m}}\\ \equiv&\sum_{1\leqslant j\leqslant k\leqslant p-1}\frac{(j^m+pmj^{m-1})(k^{2n-m}+p(2n-m)k^{2n-m-1})}{j^{2m}k^{2(2n-m)}}\\ \equiv&\sum_{1\leqslant j\leqslant k\leqslant p-1}\left(\frac{1}{j^mk^{2n-m}}+\frac{pm}{j^{m+1}k^{2n-m}}+\frac{p(2n-m)}{j^mk^{2n-m+1}}\right)\pmod{p^2}.\end{aligned}$$

Therefore, with the help of (3.1) we have

$$\begin{aligned}&\frac{2n}{2n+1}pB_{p-1-2n}-2\sum_{k=1}^{p-1}\frac{H_k^{(m)}}{k^{2n-m}}\\ \equiv&pm\sum_{k=1}^{p-1}\frac{H_k^{(m+1)}}{k^{2n-m}}+p(2n-m)\sum_{k=1}^{p-1}\frac{H_k^{(m)}}{k^{2n-m+1}}\\ \equiv&pm\frac{(-1)^m}{2n+1}\binom{2n+1}{m+1}B_{p-1-2n}+p(2n-m)\frac{(-1)^{m-1}}{2n+1}\binom{2n+1}{m}B_{p-1-2n}\\ =&(-1)^m\frac{2(m-n)}{m+1}\binom{2n+1}{m}\frac{pB_{p-1-2n}}{2n+1}\pmod{p^2}\end{aligned}$$

and hence (3.2) holds. □

Remark 3.1. By [ST, (5.4)], $\sum_{k=1}^{p-1}H_k/k^2\equiv B_{p-3}\pmod p$ for any prime $p>3$. By [M, (5)], $\sum_{k=1}^{p-1}H_k/k^3\equiv -pB_{p-5}/10\pmod{p^2}$ for any prime $p>5$. Obviously these two results are particular cases of Lemma 3.1.

Lemma 3.2. *Let $p>5$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{1-pH_k}{k^2}\equiv\frac{H_{p-1}}{p}\pmod{p^3}.\tag{3.4}$$

Proof. In view of Theorem 5.1(a) and Remark 5.1 of [S],

$$\frac{H_{p-1}^{(2)}}{2} \equiv p\left(\frac{B_{2p-4}}{2p-4} - 2\frac{B_{p-3}}{p-3}\right) \equiv -\frac{H_{p-1}}{p} \pmod{p^3}$$

and $H_{p-1}^{(3)} \equiv 0 \pmod{p^2}$. Also,

$$\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^2} = \sum_{1\leqslant j<k\leqslant p-1} \frac{1}{jk^2} \equiv -3\frac{H_{p-1}}{p^2} \pmod{p^2}$$

by [T, Theorem 2.3]. So we have

$$\sum_{k=1}^{p-1} \frac{1-pH_k}{k^2} = H_{p-1}^{(2)} - pH_{p-1}^{(3)} - p\sum_{k=1}^{p-1} \frac{H_{k-1}}{k^2} \equiv \frac{H_{p-1}}{p} \pmod{p^3}.$$

This concludes the proof. □

Proof of Theorem 1.3. Let $k \in \{1, \ldots, p-1\}$. By the proof of Theorem 1.2,

$$(-1)^{km}\binom{p/m-1}{k}^m \equiv 1 - pH_k + \frac{m-1}{2m}p^2H_k^{(2)} + p^2\sum_{1\leqslant i<j\leqslant k}\frac{1}{ij} \pmod{p^3}.$$

Thus, for any given $n \in \{1, \ldots, p-3\}$ we have

$$\begin{aligned}&\sum_{k=1}^{p-1} \frac{(-1)^{km}}{k^n}\binom{p/m-1}{k}^m \\ &\equiv \sum_{k=1}^{p-1} \frac{1-pH_k}{k^n} + \frac{m-1}{2m}p^2\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^n} + \frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2 - H_k^{(2)}}{k^n} \pmod{p^3}.\end{aligned} \tag{3.5}$$

(i) Now suppose that $1 \leqslant n \leqslant (p-3)/2$. Then $\sum_{k=1}^{p-1} H_k^{(2)}/k^{2n} \equiv 0 \pmod{p}$ since

$$2\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^n} \equiv \sum_{k=1}^{p-1}\left(\frac{H_k^{(2)}}{k^{2n}} + \frac{H_{p-k}^{(2)}}{(p-k)^{2n}}\right) \equiv \sum_{k=1}^{p-1}\frac{1/k^2}{k^{2n}} \equiv 0 \pmod{p}.$$

Thus

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n}}\binom{p/m-1}{k}^m \equiv \sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n}} \pmod{p^2} \tag{3.6}$$

and

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n}}\binom{p/m-1}{k}^m\equiv\sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n}}+\frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2}{k^{2n}}\pmod{p^3}.\quad(3.7)$$

By (3.3) and Lemma 3.1,

$$\sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n}}\equiv\left(\frac{2n}{2n+1}-1\right)pB_{p-1-2n}=-\frac{pB_{p-1-2n}}{2n+1}\pmod{p^2}.$$

So (1.7) follows from (3.6).

When $p>5$, (3.7) in the case $n=1$, together with (3.3) and the congruence

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^2}\equiv 0\pmod p$$

(cf. [Su3, (1.5)]), yields (1.6).

(ii) Fix $n\in\{1,\ldots,(p-3)/2\}$. Substituting $2n-1$ for n in (3.5) we get

$$\sum_{k=1}^{p-1}\frac{(-1)^{km}}{k^{2n-1}}\binom{p/m-1}{k}^m$$
$$\equiv\sum_{k=1}^{p-1}\frac{1-pH_k}{k^{2n-1}}+\frac{m-1}{2m}p^2\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^{2n-1}}+\frac{p^2}{2}\sum_{k=1}^{p-1}\frac{H_k^2-H_k^{(2)}}{k^{2n-1}}\pmod{p^3}.\quad(3.8)$$

In view of a known result (cf. [S, Theorem 5.1(a)]),

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\equiv\frac{n-2n^2}{2n+1}p^2B_{p-1-2n}\pmod{p^3}.$$

By Lemma 3.1,

$$\sum_{k=1}^{p-1}\frac{H_k}{k^{2n-1}}\equiv\frac{1+3n-2n^2}{2(2n+1)}pB_{p-1-2n}\pmod{p^2}$$

and

$$\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k^{2n-1}}\equiv -nB_{p-1-2n}\pmod p.$$

Note also that

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^{2n-1}}=\sum_{k=1}^{p-1}\frac{H_{p-k}^2}{(p-k)^{2n-1}}\equiv-\sum_{k=1}^{p-1}\frac{(H_k-1/k)^2}{k^{2n-1}}\pmod p$$

and hence

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^{2n-1}} \equiv \sum_{k=1}^{p-1}\frac{H_k}{k^{2n}} - \frac{1}{2}\sum_{k=1}^{p-1}\frac{1}{k^{2n+1}} \equiv \sum_{k=1}^{p-1}\frac{H_k}{k^{2n}} \equiv B_{p-1-2n} \pmod p.$$

Combining all these we obtain (1.8) from (3.8).

The proof of Theorem 1.3 is now complete. □

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