NEW CONGRUENCES FOR
CENTRAL BINOMIAL COEFFICIENTS

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Abstract. Let $p$ be a prime and let $a$ be a positive integer. In this paper
we determine $\sum_{k=0}^{p^a-1} \binom{2k}{k}/m^k$ and $\sum_{k=1}^{p^a-1} \binom{2k}{k}/(km^{k-1})$ modulo $p$ for
all $d = 0, \ldots, p^a$, where $m$ is any integer not divisible by $p$. For example,
we show that if $p \neq 2, 5$ then
$$\sum_{k=1}^{p-1} (-1)^k \frac{\binom{2k}{k}}{k} \equiv -5 \frac{F_p - (\frac{2}{p})}{p} \pmod{p},$$
where $F_n$ is the $n$th Fibonacci number and $(\cdot)$ is the Jacobi symbol. We
also prove that if $p > 3$ then
$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv \frac{8}{9} p^2 B_{p-3} \pmod{p^3},$$
where $B_n$ denotes the $n$th Bernoulli number.

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1. Introduction

A central binomial coefficient has the form \( \binom{2n}{n} \) with \( n \in \mathbb{N} = \{0, 1, \ldots\} \).

A well-known theorem of Wolstenholme (see, e.g., [5]) states that
\[
\frac{1}{2} \binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}\quad \text{for any prime } p > 3.
\]

In 2006 H. Pan and Z. W. Sun [9] used a sophisticated combinatorial identity to deduce that if \( p \) is a prime then
\[
\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \frac{p-d}{3} \pmod{p} \quad \text{for } d = 0, \ldots, p,
\]
(1.1)
where the Jacobi symbol \( \left( \frac{a}{3} \right) \) coincides with the unique integer \( \varepsilon \in \{0, \pm 1\} \) satisfying \( a \equiv \varepsilon \pmod{3} \). In a recent paper [16] the authors determined
\[
\sum_{k=0}^{p^a-1} \binom{2k}{k+d} \pmod{p^2} \quad \text{for any prime } p \text{ and } d \in \{0, 1, \ldots, p^a\} \text{ with } a \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.
\]

In this paper we extend the congruence (1.1) in a new way and derive various congruences related to recurrences. Throughout this paper, for an assertion \( A \) we set
\[
[A] = \begin{cases} 
1 & \text{if } A \text{ holds,} \\
0 & \text{otherwise.}
\end{cases}
\]

We also define two recurrences \( \{u_n(x)\}_{n \in \mathbb{N}} \) and \( \{v_n(x)\}_{n \in \mathbb{N}} \) of polynomials as follows:
\[
u_0(x) = 0, \quad u_1(x) = 1, \quad \text{and } u_{n+1}(x) = xu_n(x) - u_{n-1}(x) \quad (n = 1, 2, \ldots),
\]
and
\[
v_0(x) = 2, \quad v_1(x) = x, \quad \text{and } v_{n+1}(x) = xv_n(x) - v_{n-1}(x) \quad (n = 1, 2, \ldots).
\]

For a fixed integer \( x \), the sequences \( \{u_n(x)\}_{n \in \mathbb{N}} \) and \( \{v_n(x)\}_{n \in \mathbb{N}} \) are linear recurrences of integers. By induction, for any \( n \in \mathbb{N} \) we have
\[
u_n(-x) = (-1)^{n-1} u_n(x) \quad \text{and } \quad v_n(-x) = (-1)^n v_n(x).
\]

(1.2)

Now we state our first theorem.

**Theorem 1.1.** Let \( p \) be a prime and let \( d \in \{0, \ldots, p^a\} \) with \( a \in \mathbb{Z}^+ \). Let \( m \in \mathbb{Z} \) with \( p \nmid m \). Then we have
\[
\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k+d}}{m^k} \equiv u_{p^a-d}(m-2) \pmod{p}
\]
(1.3)
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\[ d \sum_{k=1}^{p^a-1} \frac{(2k)}{k+d} \equiv 2(-1)^d + v_{p^a-d}(m-2) \pmod{p} \quad \text{provided } d > 0. \quad (1.4) \]

If \( p \neq 2 \), then

\[ d \sum_{k=0}^{p^a-1} \frac{(2k)}{k+d} \equiv -u_{p^a-d(m-4)}(m-2) \pmod{p} \quad (1.5) \]

and also

\[ d \sum_{k=1}^{p^a-1} \frac{(2k)}{k+d} \equiv 2(-1)^d + v_{d(m-4)}(m-2) \pmod{p} \quad \text{provided } d > 0, \quad (1.6) \]

where \( u_{-1}(x) = xu_0(x) - u_1(x) = -1 \) and \( v_{-1}(x) = xv_0(x) - v_1(x) = x \).

Remark 1.1. Let \( p \) be any prime and let \( a \in \mathbb{Z}^+ \). As \( u_{n(1)} = \left( \frac{a}{3} \right) \) for \( n = 0,1,2,\ldots, (1.3) \) in the case \( m = 1 \) yields that

\[ d \sum_{k=0}^{p^a-1} \frac{(2k)}{k+d} \equiv \left( \frac{p^a-d}{3} \right) \pmod{p} \quad \text{for every } d = 0,\ldots,p^a. \]

Since \( v_n(-1) = 3 [3 \mid n] - 1 \) for all \( n \in \mathbb{N} \), by (1.4) in the case \( m = 1 \), for \( d \in \{1,\ldots,p^a\} \) we have

\[ d \sum_{k=1}^{p^a-1} \frac{(2k)}{k+d} \equiv \begin{cases} 2(-1)^d + 2 \pmod{p} & \text{if } p^a \equiv d \pmod{3}, \\ 2(-1)^d - 1 \pmod{p} & \text{otherwise.} \end{cases} \]

The well-known Fibonacci sequence \( \{F_n\}_{n \in \mathbb{N}} \) is defined by

\[ F_0 = 0, \quad F_1 = 1, \quad \text{and } F_{n+1} = F_n + F_{n-1} \text{ for } n = 1,2,3,\ldots. \]

Its companion \( \{L_n\}_{n \in \mathbb{N}} \), the Lucas sequence, is given by

\[ L_0 = 2, \quad L_1 = 1, \quad \text{and } L_{n+1} = L_n + L_{n-1} \text{ for } n = 1,2,3,\ldots. \]

Define

\[ F_{-1} = F_1 - F_0 = 1, \quad F_{-2} = F_0 - F_{-1} = -1, \quad L_{-1} = L_1 - L_0 = -1, \quad L_{-2} = L_0 - L_{-1} = 3. \]

By induction, \( F_{2n} = u_n(3) \) and \( L_{2n} = v_n(3) \) for \( n = -1,0,1,\ldots. \) Note also that \( u_{2n}(0) = v_{2n+1}(0) = 0 \) and \( v_{2n}(0)/2 = u_{2n+1}(0) = (-1)^n \) for all \( n \in \mathbb{N} \). Thus, with the help of (1.2), Theorem 1.1 in the cases \( m = -1,2 \) gives the following consequence.
Corollary 1.1. Let $p$ be an odd prime and let $d \in \{0, 1, \ldots, p^a\}$ with $a \in \mathbb{Z}^+$. Then
\begin{equation}
\sum_{k=0}^{p^a - 1} (-1)^k \binom{2k}{k+d} \equiv (-1)^{d-[p\not\equiv 5]} F_{2(d-(\frac{p}{2n}))} (\mod p), \tag{1.7}
\end{equation}
and
\begin{equation}
d \sum_{k=1}^{p^a - 1} (-1)^k \frac{2k}{k+d} \equiv (-1)^d L_{2(d-(\frac{p}{2n}))} - 2(-1)^d (\mod p) \tag{1.8}
\end{equation}
provided $d > 0$. Also,
\begin{equation}
\sum_{k=0}^{p^a - 1} \frac{2k}{k+d} \equiv \begin{cases} 0 \ (\mod p) & \text{if } p^a \equiv d \ (\mod 2), \\ 1 \ (\mod p) & \text{if } p^a \equiv d+1 \ (\mod 4), \\ -1 \ (\mod p) & \text{if } p^a \equiv d-1 \ (\mod 4), \end{cases} \tag{1.9}
\end{equation}
and for $d > 0$ we have
\begin{equation}
d \sum_{k=1}^{p^a - 1} \frac{2k}{k+d} - (-1)^d \equiv \begin{cases} 0 \ (\mod p) & \text{if } p^a \not\equiv d \ (\mod 2), \\ 1 \ (\mod p) & \text{if } p^a \equiv d \ (\mod 4), \\ -1 \ (\mod p) & \text{if } p^a \equiv d+2 \ (\mod 4). \end{cases} \tag{1.10}
\end{equation}

Our following result can be viewed as a complement to Theorem 1.1.

Theorem 1.2. Let $p$ be a prime and let $m$ be an integer not divisible by $p$. Then we have
\begin{equation}
\frac{1}{2} \sum_{k=1}^{p-1} (-1)^k \binom{2k}{km^{p-1}-1} \equiv \frac{m^p - V_p(m)}{p} \ (\mod p), \tag{1.11}
\end{equation}
where the polynomial sequence $\{V_n(x)\}_{n \in \mathbb{N}}$ is defined as follows:
\begin{equation}
V_0(x) = 2, \ V_1(x) = x, \ \text{and} \ V_{n+1}(x) = x(V_n(x) + V_{n-1}(x)) \ (n \in \mathbb{Z}^+). \tag{1.12}
\end{equation}

Given a prime $p$ and an integer $a$ not divisible by $p$, we use $q_p(a)$ to denote the integer $(a^{p-1} - 1)/p$ and call $q_p(a)$ a Fermat quotient with base $a$. See E. Lehmer [7] for connections between Fermat quotients and Fermat’s last theorem.
Corollary 1.2. Let \( p \) be an odd prime. Then
\[
\sum_{k=1}^{p-1} \binom{2k}{k} \equiv \sum_{k=1}^{p-1} \binom{2k}{k} k^{-k} \equiv 2q_p(2) \pmod{p}.
\]
If \( p \neq 3 \) then
\[
\sum_{k=1}^{p-1} \binom{2k}{k} k^{-k} \equiv 3q_p(3) \pmod{p}.
\]

Corollary 1.3. Let \( p \) be an odd prime. Then
(i) If \( p \neq 5 \), we have
\[
\sum_{k=1}^{p-1} \binom{2k}{k} k^{-k} \equiv -5q_p(2) - \binom{2p}{p} S_{p-(\frac{p}{3})} \pmod{p}
\]
(ii) Define the Pell sequence \( \{P_n\}_{n \in \mathbb{N}} \) by
\[
P_0 = 0, \ P_1 = 1, \ \text{and} \ \ P_{n+1} = 2P_n + P_{n-1} (n = 1, 2, 3, \ldots).
\]

Then
\[
\sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} k^{-k} \equiv -5q_p(2) - \frac{p}{p} S_{P-(\frac{p}{3})} \pmod{p}.
\]

(iii) Let \( \{S_n\}_{n \in \mathbb{N}} \) be the sequence defined by
\[
S_0 = 0, \ S_1 = 1, \ \text{and} \ \ S_{n+1} = 4S_n - S_{n-1} (n = 1, 2, 3, \ldots).
\]

If \( p > 3 \), then
\[
\sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} k^{-k} \equiv q_p(2) - 6 \frac{2}{p} S_{\frac{p-(\frac{p}{3})}{2}} \equiv \sum_{0<k<5p/6} (-1)^{k-1} \pmod{p}
\]
and
\[
\sum_{k=1}^{p-1} \binom{2k}{k} k^{-k} \equiv q_p(2) + q_p(3) - 2 \frac{2}{p} S_{\frac{p-(\frac{p}{3})}{2}} \pmod{p}.
\]
Remark 1.2. (a) A prime \( p \neq 2, 5 \) is called a Wall-Sun-Sun prime if 
\[ F_p(\xi) \equiv 0 \pmod{p^2} \] (cf. [1]). In 1992 Z. H. Sun and Z. W. Sun [13]
showed that Fermat’s equation \( x^p + y^p = z^p \) has no integer solutions satisfying \( p \nmid xyz \)
unless \( p \) is a Wall-Sun-Sun prime. There are no Wall-Sun-Sun primes below \( 2 \times 10^{14} \) (cf. [8]). In 1982 H. C. Williams [10] showed that 
\[ F_p(\xi) - (p^5) \equiv 0 \pmod{p^2} \] (1.18).

(b) The second congruences in (1.17) and (1.18) are essentially due to Z.
W. Sun [14, 15]. For other information about the sequence \( \{S_n\}_{n \in \mathbb{N}} \) the
reader may consult [11].

In 2006 Pan and Sun [9] proved that 
\[ p^a - 1 \sum_{k=0}^{p^a-1} \binom{2k}{k} x^{n-1-k} + [d > 0]x^n u_d(x - 2) \]
for any prime \( p > 3 \). Here we determine the sum modulo \( p^3 \).

**Theorem 1.3.** Let \( p \) be any prime and let \( a \in \mathbb{Z}^+ \). Then we have

\[ p^a - 1 \sum_{k=1}^{p^a-1} \binom{2k}{k} \equiv \begin{cases} 
2 \pmod{p^3} & \text{if } p = 2, \\
5 \pmod{p^3} & \text{if } p = 3, \\
\frac{8}{3} p^2 B_{p-3} \pmod{p^3} & \text{otherwise},
\end{cases} \] (1.20)

where \( B_0, B_1, B_2, \ldots \) are the well-known Bernoulli numbers.

The following conjecture, which is related to (1.7) in the case \( d = 0 \),
seems very challenging.

**Conjecture 1.1.** Let \( p \neq 2, 5 \) be a prime and let \( a \in \mathbb{Z}^+ \). Then

\[ p^a - 1 \sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left( \frac{p^a}{5} \right) \left( 1 - 2F_{p^a - (\xi)} \right) \pmod{p^3}. \]

In the next section we are going to present two auxiliary identities. Theorem 1.1, Theorem 1.2 and Corollaries 1.2-1.3, and Theorem 1.3 will
be proved in Sections 3, 4 and 5 respectively.

2. An auxiliary theorem

**Theorem 2.1.** For any \( n \in \mathbb{Z}^+ \) and \( d \in \mathbb{Z} \), we have

\[ \sum_{0 \leq k < n} \binom{2k}{k+d} x^{n-1-k} + [d > 0]x^n u_d(x - 2) \]

\[ = \sum_{0 \leq k < n+d} \binom{2n}{k} u_{n+d-k}(x - 2) \] (2.1)
and
\[
d \sum_{0 \leq k < n} \binom{2k}{k} x^{n-k} - [d \geq 0] x^n v_d(x-2) + [d = 0] x^n
= - \sum_{0 \leq k < n+d} \binom{2n}{k} v_{n+d-k} (x-2) - 2 \binom{2n-1}{n+d-1}.
\] (2.2)

Proof. (i) We use induction on \( n \in \mathbb{Z}^+ \) to prove (2.1).

Since \((x-2)u_d(x-2) = u_{d+1}(x-2) + u_{d-1}(x-2) \) for \( d = 1, 2, 3, \ldots \),
we can easily see that (2.1) with \( n = 1 \) holds for all \( d \in \mathbb{Z} \).

Now fix \( n \in \mathbb{Z}^+ \) and assume (2.1) for all \( d \in \mathbb{Z} \). Let \( d \) be any integer.
For \( k \in \mathbb{N} \), it is easy to see that
\[
\binom{2n+2}{k} = \binom{2n}{k} + 2 \binom{2n}{k-1} + \binom{2n}{k-2}.
\]

Thus,
\[
\sum_{0 \leq k < (n+1)+d} \binom{2n+2}{k} u_{n+1+d-k} (x-2)
= \sum_{0 \leq k < n+(d+1)} \binom{2n}{k} u_{n+(d+1)-k} (x-2)
+ 2 \sum_{0 \leq j < n+d} \binom{2n}{j} u_{n+d-j} (x-2)
+ \sum_{0 \leq i < n+(d-1)} \binom{2n}{i} u_{n+(d-1)-i} (x-2).
\]

By the induction hypothesis, for any \( r \in \mathbb{Z} \) we have
\[
\sum_{0 \leq k < n+r} \binom{2n}{k} u_{n+r-k} (x-2) = \sum_{0 \leq k < n} \binom{2k}{k} x^{n-1-k} + [r > 0] x^n u_r(x-2),
\]
So, from the above we get

\[
\sum_{0 \leq k < (n+1)+d} \binom{2n+2}{k} u_{n+1+d-k}(x-2)
\]

\[
= \sum_{0 \leq k < n} \left( \binom{2k}{k+d+1} + 2 \binom{2k}{k+d} + \binom{2k}{k+d-1} \right) x^{n-1-k}
\]

\[
+ [d \geq 0]x^n u_{d+1}(x-2) + 2[d \geq 0]x^n u_d(x-2) + [d > 0]x^n u_{d-1}(x-2)
\]

\[
= \sum_{0 \leq k < n} \left( \binom{2k+1}{k+d+1} + \binom{2k+1}{k+d} \right) x^{n-1-k} - [d = 0]x^n u_{-1}(x-2)
\]

\[
+ [d \geq 0]x^n (u_{d+1}(x-2) + 2u_d(x-2) + u_{d-1}(x-2))
\]

\[
= \sum_{0 \leq k < n} \left( \frac{2k+1}{k+d+1} \right) x^{n-1-k} + [d = 0]x^n + [d \geq 0]x^n x u_d(x-2)
\]

\[
= \sum_{0 \leq k < n+1} \left( \frac{2k}{k+d} \right) x^{(n+1)-1-k} + [d > 0]x^n u_d(x-2).
\]

This concludes the induction step and hence (2.1) holds.

(ii) By induction, \( v_k(x-2) = 2u_{k+1}(x-2) - (x-2)u_k(x-2) \) for all \( k \in \mathbb{Z} \). Thus, with the help of (2.1), we have

\[
\sum_{0 \leq k \leq n+d} \binom{2n}{k} v_{n+d-k}(x-2)
\]

\[
= 2 \sum_{0 \leq k < n+1+d} \binom{2n}{k} u_{n+d+1-k}(x-2)
\]

\[
- (x-2) \sum_{0 \leq k < n+d} \binom{2n}{k} u_{n+d-k}(x-2)
\]

\[
= 2 \sum_{0 \leq k < n} \binom{2k}{k+d+1} x^{n-1-k} + [d+1 > 0]x^n 2u_{d+1}(x-2)
\]

\[
- (x-2) \left( \sum_{0 \leq k < n} \binom{2k}{k+d} x^{n-1-k} + [d > 0]x^n u_d(x-2) \right)
\]

\[
= \sum_{0 \leq k < n} \left( \frac{2k}{k+d+1} - (x-2) \frac{2k}{k+d} \right) x^{n-1-k} + [d \geq 0]x^n u_d(x-2).
\]

For \( k \in \mathbb{Z}^+ \) we have

\[
\frac{2k-2}{k+d} + \frac{2k-2}{k+d-1} = \frac{2k-1}{k+d} = \frac{2k-1}{k-d-1}
\]

\[
= \frac{k-d}{2k} \cdot \frac{2k}{k+d} = \frac{k-d}{2k} \cdot \frac{2k}{k+d} = \frac{1}{2} \left( \frac{2k}{k+d} - \frac{d}{2k} \frac{2k}{k+d} \right).
\]
Thus
\[
\frac{1}{2} \sum_{0 < k < n} \binom{2k}{k} x^{n-k} - \frac{d}{2} \sum_{0 < k < n} \binom{2k}{k+d} x^{n-k} = \sum_{0 < k \leq n} \left( \binom{2k - 2}{k + d} + \binom{2k - 2}{k + d - 1} \right) x^{n-k} - \binom{2n - 2 + 1}{n + d}.
\]

It follows that
\[
d \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{k} x^{n-k} \left[ d = 0 \right] x^n - 2 \binom{2n - 1}{n + d} = \sum_{0 \leq k < n} \left( x - 2 \right) \binom{2k}{k + d} - 2 \binom{2k}{k + d + 1} x^{n-1-k}.
\]

Combining the above we obtain
\[
\sum_{0 \leq k \leq n + d} \binom{2n}{k} v_{n+d-k}(x-2) - \left[ d \geq 0 \right] x^n v_d(x-2) = -d \sum_{0 < k < n} \frac{\binom{2k}{k+d}}{k} x^{n-k} \left[ d = 0 \right] x^n + 2 \binom{2n - 1}{n + d},
\]
from which (2.2) follows.

\[\square\]

**Corollary 2.1.** Let \( n \in \mathbb{Z}^+ \) and \( d \in \mathbb{N} \). Then

\[
\sum_{0 \leq k \leq n} \binom{2k}{k} + \binom{d}{3} = \sum_{0 \leq k < n+d} \binom{2n}{k} \left( n + d - k \right) / 3, \quad (2.3)
\]

\[
\sum_{0 \leq k < n} (-1)^{k+d} \binom{2k}{k+d} + F_{2d} = \sum_{0 \leq k < n+d} (-1)^k \binom{2n}{k} F_{2(n+d-k)}, \quad (2.4)
\]

and

\[
d \sum_{0 < k < n} \frac{(-1)^{k+d} \binom{2k}{k+d}}{k} + \sum_{0 < k < n+d} \binom{2n}{k} (-1)^k L_{2(n+d-k)} = L_{2d} - (-1)^{n+d} \binom{2n - 1}{n + d - 1} - \left[ d = 0 \right]. \quad (2.5)
\]

**Proof.** For \( j \in \mathbb{N} \) we have \( u_j(-1) = \left( \frac{1}{2} \right) \), \( (-1)^{j-1} u_j(-3) = u_j(3) = F_{2j} \) and \( (-1)^j v_j(-3) = v_j(3) = L_{2j} \). Thus, (2.1) in the case \( x = 1 \) yields (2.3), and (2.1) and (2.2) in the case \( x = -1 \) reduce to (2.4) and (2.5) respectively. This concludes the proof. \( \square \)
3. Proof of Theorem 1.1

Given $A, B \in \mathbb{Z}$ we define the Lucas sequence $u_n = u_n(A, B)$ ($n \in \mathbb{N}$) and its companion $v_n = v_n(A, B)$ ($n \in \mathbb{N}$) as follows:

$$u_0 = 0, \ u_1 = 1, \text{ and } u_{n+1} = Au_n - Bu_{n-1} \text{ for } n = 1, 2, 3, \ldots,$$

and

$$v_0 = 2, \ v_1 = A, \text{ and } v_{n+1} = Av_n - Bv_{n-1} \text{ for } n = 1, 2, 3, \ldots.$$

It is well known that

$$u_n = \sum_{0 \leq k < n} \alpha^k \beta^{n-1-k} \text{ and } v_n = \alpha^n + \beta^n \text{ for all } n \in \mathbb{N},$$

where $\alpha$ and $\beta$ are the two roots of the equation $x^2 - Ax + B = 0$. It follows that if $n \in \mathbb{N}$ and $m \in \{n, n+1, \ldots\}$ then

$$Au_n + v_n = 2u_{n+1} \text{ and } u_n v_n - u_{n+1} v_m = 2B^n u_{m-n}.$$

**Lemma 3.1.** Let $A, B \in \mathbb{Z}$ with $B \neq 0$. Let $u_n = u_n(A, B)$ for $n \in \mathbb{N}$, and define $u_{-1} = (u_1 - Au_0)/(-B) = -1/B$. Let $p$ be an odd prime, and let $a \in \mathbb{Z}^+$ and $d \in \{0, 1, \ldots, p^a\}$. Then we have

$$B^d u_{pa-d} \equiv -c(A, B) u_d \pmod{p}, \quad (3.1)$$

where $\Delta = A^2 - 4B$ and

$$c(A, B) = \begin{cases} A/2 & \text{if } p \mid \Delta, \\ B & \text{if } (\Delta / p^a) = 1, \\ 1 & \text{if } (\Delta / p^a) = -1. \end{cases}$$

**Proof.** The two roots of the equation $x^2 - Ax + B = 0$ are algebraic integers $\alpha = (A + \sqrt{\Delta})/2$ and $\beta = (A - \sqrt{\Delta})/2$. Since

$$\binom{p^a}{k} = \frac{p^a}{k} \binom{p^a - 1}{k-1} \equiv 0 \pmod{p} \text{ for } k = 1, \ldots, p^a - 1,$$

we have

$$v_{p^a} = \alpha^{p^a} + \beta^{p^a} \equiv (\alpha + \beta)^{p^a} = A^{p^a} \equiv A^{p^a-1} \equiv \cdots \equiv A \pmod{p}$$
with the help of Fermat’s little theorem. If $\Delta \neq 0$, then

$$u_{p^a} = \frac{\alpha^{p^a} - \beta^{p^a}}{\alpha - \beta} = \frac{1}{\sqrt{\Delta}} \left( \left( \frac{A + \sqrt{\Delta}}{2} \right)^{p^a} - \left( \frac{A - \sqrt{\Delta}}{2} \right)^{p^a} \right)$$

$$= \frac{1}{2p^a \sqrt{\Delta}} \sum_{k=0}^{p^a} \binom{p^a}{k} A^{p^a - k} \left( (\sqrt{\Delta})^k - (-\sqrt{\Delta})^k \right)$$

$$= \frac{1}{2p^a - 1} \sum_{2k-1}^{p^a} \binom{p^a}{k} A^{p^a - k} \Delta^{(k-1)/2};$$

if $\Delta = 0$ then $\alpha = \beta = A/2$ and hence $u_{p^a} = p^a (A/2)^{p^a - 1}$. So we always have

$$u_{p^a} = \frac{1}{2p^a - 1} \sum_{2k-1}^{p^a} \binom{p^a}{k} A^{p^a - k} \Delta^{(k-1)/2}.$$

Note that $2^{p^a - 1} \equiv 1 \pmod{p}$ by Fermat’s little theorem. Thus, by Euler’s criterion,

$$u_{p^a} \equiv \left( \frac{p^a}{p^a} \right) \Delta^{(p^a - 1)/2} = \left( \Delta^{(p-1)/2} \right)^{\sum_{k=0}^{p^a-1} p^k} \equiv \left( \frac{\Delta}{p} \right)^a \equiv \left( \frac{\Delta}{p^a} \right) \pmod{p}.$$

Observe that

$$2B^d u_{p^a - d} = u_{p^a} v_d - u_d v_{p^a} \equiv \left( \frac{\Delta}{p^a} \right) v_d - u_d A \pmod{p}.$$

When $p \mid \Delta$, this yields

$$B^d u_{p^a - d} \equiv -\frac{A}{2} u_d \pmod{p}.$$

If $\left( \frac{\Delta}{p^a} \right) = 1$, then

$$2B^d u_{p^a - d} \equiv v_d - Au_d = 2(u_{d+1} - Au_d) = -2B u_{d-1} \pmod{p}$$

and hence $B^d u_{p^a - d} \equiv -Bu_{d-1} \pmod{p}$. If $\left( \frac{\Delta}{p^a} \right) = -1$, then

$$2B^d u_{p^a - d} \equiv -v_d - Au_d = -2u_{d+1} \pmod{p}$$

and thus $B^d u_{p^a - d} \equiv -u_{d+1} \pmod{p}$. So (3.1) follows. \(\square\)
Proof of Theorem 1.1. For $n = -1, 0, 1, \ldots$ let $u_n = u_n(m-2)$ and $v_n = v_n(m-2)$.

By Theorem 2.1,

$$\sum_{k=0}^{p^n-1} \frac{2k}{k-d} m^{p^n-1-k} = \sum_{0 \leq k < p^n-d} \binom{2p^n}{k} u_{p^n-d-k};$$

also, for $d > 0$ we have

$$-d \sum_{0 < k < p^n} \frac{2k}{k} m^{p^n-k} = - \sum_{0 < k < p^n-d} \binom{2p^n}{k} v_{p^n-d-k} - 2 \binom{2p^n-1}{p^n-d-1}.$$

By Fermat’s little theorem, $m^{p^n} \equiv m \pmod{p}$. For $k \in \{1, \ldots, p^n-1\}$ clearly

$$\binom{2p^n}{k} = \binom{2p^n}{k} \binom{2p^n-1}{k-1} \equiv 0 \pmod{p};$$

also, if $d < p^n$ then

$$\binom{2p^n-1}{p^n-d-1} = \prod_{0 < j < p^n-d} \binom{2p^n}{j} \equiv (-1)^{p^n-d-1} \equiv (-1)^d \pmod{p}.$$

Therefore

$$\sum_{k=0}^{p^n-1} \binom{2k}{k+d} m^k \equiv [d \neq p^n] \binom{2p^n}{0} u_{p^n-d} = u_{p^n-d} \pmod{p};$$

if $d > 0$ then

$$d \sum_{k=1}^{p^n-1} \binom{2k}{k+d} m^{k-1} \equiv [d \neq p^n] \binom{2p^n}{0} v_{p^n-d} + 2[d \neq p^n](-1)^d$$

$$\equiv v_{p^n-d} + 2(-1)^d \pmod{p}.$$

So we have (1.3) and (1.4).

Now assume $p \neq 2$ and set $\Delta = (m-2)^2 - 4 \times 1 = m(m-4)$. As $p \nmid m$, if $p \mid \Delta$ then $m \equiv 4 \pmod{p}$ and hence $(m-2)/2 \equiv 1 \pmod{p}$. Thus, with the help of Lemma 3.1, we have

$$\sum_{k=0}^{p^n-1} \binom{2k}{k+d} m^k \equiv u_{p^n-d} \equiv -u_{d-(d\Delta)} \pmod{p},$$

which proves (1.5). If $d > 0$, then

$$v_{d-(d\Delta)} = 2u_{d-(d\Delta)+1} - (m-2)u_{d-(d\Delta)}$$

$$= -2u_{d-1-(d\Delta)} + (m-2)u_{d-(d\Delta)}$$

$$\equiv 2u_{p^n-d+1} - (m-2)u_{p^n-d} \equiv v_{p^n-d} \pmod{p}.$$

Thus (1.6) follows from (1.4). We are done. \(\square\)
4. Proofs of Theorem 1.2 and Corollaries 1.2-1.3

Lemma 4.1. For any positive integer \( n \), we have

\[
\frac{1}{2} \sum_{0 < k < n} \binom{2k}{k} x^k = \sum_{0 < d < n} (-1)^{d-1} \sum_{0 < k < n} \binom{2k}{k+d} x^k.
\]  

(4.1)

Proof. Observe that

\[
\sum_{d=0}^{n-1} (-1)^d \sum_{0 < k < n} \binom{2k}{k+d} x^k = \sum_{0 < k < n} \frac{1}{k(-x)^k} \sum_{d=0}^{n-1} (-1)^{k+d} \binom{2k}{k+d}.
\]

\[
= \sum_{0 < k < n} \frac{1}{2k(-x)^k} \sum_{j=0}^{2k} (-1)^j \left( \begin{array}{c} 2k \\ j \end{array} \right) + (-1)^{2k-j} \binom{2k}{2k-j}.
\]

\[
= \sum_{0 < k < n} \frac{1}{2k(-x)^k} \left( (1 - 1)^{2k} + (-1)^{k} \binom{2k}{k} \right) = \frac{1}{2} \sum_{0 < k < n} \binom{2k}{k} x^k.
\]

So (4.1) follows. □

Proof of Theorem 1.2. By Lemma 4.1,

\[
\frac{1}{2} \sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} \frac{1}{km(k-1)} = \sum_{d=1}^{p-1} (-1)^{d-1} \sum_{k=1}^{p-1} \binom{2k}{k+d} \frac{1}{k(-m)^{k-1}}.
\]

In view of (1.4) and the basic fact

\[
\frac{1}{p} \binom{p}{d} = \frac{1}{d} \prod_{0 < k < d} \frac{p - k}{k} \equiv \frac{(-1)^{d-1}}{d} \mod p \quad (d = 1, \ldots, p-1),
\]

we have

\[
\sum_{d=1}^{p-1} (-1)^d \sum_{k=1}^{p-1} \binom{2k}{k+m} \frac{1}{k(-m)^{k-1}}
\]

\[
\equiv \sum_{d=1}^{p-1} (-1)^d \frac{1}{d} v_{p-d}(-m-2) + 2(-1)^d
\]

\[
\equiv \sum_{d=1}^{p-1} (-1)^d \frac{1}{d} v_{p-d}(-m-2) + \sum_{d=1}^{p-1} \left( \frac{1}{d} + \frac{1}{p-d} \right)
\]

\[
\equiv -\frac{1}{p} \sum_{d=1}^{p-1} \binom{p}{d} v_{p-d}(-m-2) = -\frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} v_k(-m-2) \mod p.
\]
Let $\alpha$ and $\beta$ be the two roots of the equation $x^2 - mx - m = 0$. Then
$(-\alpha - 1) + (-\beta - 1) = -m - 2$ and $(-\alpha - 1)(-\beta - 1) = 1$, also
$$V_p(m) = \alpha^p + \beta^p \equiv (\alpha + \beta)^p = m^p \equiv m \pmod{p}.$$ 

In the case $p \neq 2$, we have
$$\sum_{k=1}^{p-1} \binom{p}{k} v_k(-m+2) = \sum_{k=1}^{p-1} \binom{p}{k}((-\alpha - 1)^k + (-\beta - 1)^k)$$
$$=(-\alpha)^p + (-\beta)^p - (\alpha - 1)^p - (\beta - 1)^p$$
$$= - V_p(m) + \frac{(\alpha^p + \beta^p)^2}{m^p} = \left(1 + \frac{V_p(m) - m^p}{m^p}\right)(V_p(m) - m^p)$$
$$\equiv V_p(m) - m^p \pmod{p^2} \quad (\text{since } V_p(m) \equiv m^p \pmod{p}).$$

Note also that
$$\sum_{k=1}^{2-1} \binom{2}{k} v_k(-m+2) = 2(-m+2) \equiv 2m = V_2(m) - m^2 \pmod{2^2}.$$ 

Therefore (1.11) follows from the above. $\square$

**Proof of Corollary 1.2.** By induction, whenever $n \in \mathbb{N}$ we have
$$V_{4n}(-2) = (-1)^n 2^{2n+1}, \quad V_{4n+1}(-2) = (-1)^{n+1} 2^{2n+1},$$
$$V_{4n+2}(-2) = 0, \quad V_{4n+3}(-2) = (-1)^n 2^{2n+2}.$$ 

It follows that
$$V_p(-2) = - \left(\frac{2}{p}\right) 2^{(p+1)/2}.$$ 

Combining this with (1.11) in the case $m = -2$, we get
$$\sum_{k=1}^{p-1} \binom{2k}{k} \frac{V_p(-2) - (-2)^p}{p} = 2^{(p-1)/2} \frac{2^{(p-1)/2} - \left(\frac{2}{p}\right)}{p}$$
$$\equiv \left(2^{(p-1)/2} + \left(\frac{2}{p}\right)\right) \frac{2^{(p-1)/2} - \left(\frac{2}{p}\right)}{p} = q_p(2) \pmod{p}.$$ 

By induction, $V_n(-4) = (-1)^n 2^{n+1}$ for all $n \in \mathbb{N}$. Thus, by (1.11) with $m = -4$, we have
$$\frac{1}{2} \sum_{k=1}^{p-1} \binom{2k}{k} \frac{V_p(-4) - (-4)^p}{p} = 2^{p} \frac{2^p - 2}{p} \equiv 4q_p(2) \pmod{p}.$$
Therefore (1.12) holds.

Now assume that \( p \neq 3 \). By induction, for \( n \in \mathbb{N} \) we have

\[
V_n(-3) = \begin{cases} 
(3[3 \mid n] - 1)(-3)^{n/2} & \text{if } 2 \mid n, \\
\left(\frac{3}{2}\right)(-3)^{(n+1)/2} & \text{if } 2 \nmid n.
\end{cases}
\]

Applying (1.11) with \( m = -3 \) we get

\[
\frac{1}{2} \sum_{k=1}^{p-1} \binom{2k}{k} k^{3k-1} \equiv \frac{V_p(-3) - (-3)^p}{p} = \frac{(-3)^{(p+1)/2}(-3)^{(p-1)/2} - \left(\frac{-3}{p}\right)}{p}
\]

\[
\equiv \frac{3}{2} \left(-1\right)^{(p-1)/2} + \left(\frac{-3}{p}\right) \left(-3\right)^{(p-1)/2} - \left(\frac{-3}{p}\right)
\]

\[
\equiv \frac{3}{2} \left(-3\right)^{p-1} - 1 = \frac{3}{2} q_p(3) \pmod{p}.
\]

So (1.13) is valid. □

Proof of Corollary 1.3. (i) Applying Theorem 1.2 with \( m = 1 \), we obtain that

\[
\frac{1}{2} \sum_{k=1}^{p-1} (-1)^k \binom{2k}{k} k \equiv \frac{1 - L_p}{p} \pmod{p}.
\]

Let \( \alpha \) and \( \beta \) be the two roots of the equation \( x^2 - x - 1 = 0 \). Suppose \( p \neq 5 \) and set \( n = (p - (\frac{p}{5}))/2 \). It is known that

\[
L_n^2 - 5F_n^2 = (\alpha^n + \beta^n)^2 - (\alpha - \beta)^2 \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 = 4(\alpha\beta)^n = 4(-1)^n
\]

and

\[
L_{2n} = \alpha^{2n} + \beta^{2n} = (\alpha^n + \beta^n)^2 - 2(\alpha\beta)^n = L_n^2 - 2(-1)^n.
\]

By [13, Corollary 1], \( p \mid F_n \) if \( p \equiv 1 \pmod{4} \), and \( p \mid L_n \) if \( p \equiv 3 \pmod{4} \). Thus

\[
L_{p-\left(\frac{p}{5}\right)} = L_{2n} = 5F_n^2 + 2(-1)^n = L_n^2 - 2(-1)^n \equiv 2 \left(\frac{p}{5}\right) \pmod{p^2}.
\]

By induction,

\[
2L_k = 5F_{k-1} + L_{k-1} = 5F_{k+1} - L_{k+1} \text{ for } k = 1, 2, 3, \ldots.
\]

Therefore

\[
2L_p = 5F_{p-\left(\frac{p}{5}\right)} + \left(\frac{p}{5}\right) L_{p-\left(\frac{p}{5}\right)} \equiv 5F_{p-\left(\frac{p}{5}\right)} + 2 \pmod{p^2}.
\]
and hence

\[
\sum_{k=1}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k} \equiv -2 \frac{L_p - 1}{p} \equiv -5 \frac{F_{p-(\frac{p}{5})}}{p} \pmod{p}.
\]

This proves (1.14).

By (1.11) in the case \(m = 5\),

\[
\sum_{k=1}^{\frac{p-1}{2}} (-1)^k \binom{2k}{k} \equiv \frac{5^p - V_p(5)}{p} \pmod{p}.
\]

Since \((5 + 3\sqrt{5})/2\) and \((5 - 3\sqrt{5})/2\) are the two roots of the equation \(x^2 - 5x - 5 = 0\),

\[
V_p(5) = \left(\frac{5 + 3\sqrt{5}}{2}\right)^p + \left(\frac{5 - 3\sqrt{5}}{2}\right)^p
\]

\[
= \sqrt{5}^p \left[ \left(\frac{1 + \sqrt{5}}{2}\right)^{2p} - \left(\frac{1 - \sqrt{5}}{2}\right)^{2p} \right]
\]

\[
= 5^{(p+1)/2} \frac{\alpha^p - \beta^p}{\alpha - \beta} (\alpha^p + \beta^p) = 5^{(p+1)/2} F_p L_p.
\]

As

\[
L_p \equiv 1 + \frac{5}{2} F_{p-(\frac{p}{5})} \pmod{p^2}
\]

and

\[
L_p = F_p + 2F_{p-1} = 2F_{p+1} - F_p = 2F_{p-(\frac{p}{5})} + \left(\frac{p}{5}\right) F_p,
\]

we have

\[
\left(\frac{p}{5}\right) F_p L_p = L_p (L_p - 2F_{p-(\frac{p}{5})})
\]

\[
\equiv \left(1 + \frac{5}{2} F_{p-(\frac{p}{5})}\right) \left(1 + \frac{1}{2} F_{p-(\frac{p}{5})}\right) \equiv 1 + 3F_{p-(\frac{p}{5})} \pmod{p^2}
\]

and hence

\[
V_p(5) = 5^{(p+1)/2} F_p L_p
\]

\[
\equiv 5^{(p+1)/2} \left(\frac{5}{p}\right) (1 + 3F_{p-(\frac{p}{5})}) \equiv 5^{(p+1)/2} \left(\frac{5}{p}\right) + 15 F_{p-(\frac{p}{5})} \pmod{p^2}.
\]
Therefore
\[
\sum_{k=1}^{p-1} \frac{(-1)^k \binom{2k}{k}}{k5^k} \equiv \frac{2}{5} \frac{5^p - 5(p+1)/2\left(\frac{5}{p}\right)}{p} - 15F_{p-(\xi)} \\left(\frac{5}{p}\right) - 6 \frac{F_{p-(\xi)}}{p} \equiv q_p(5) - 6 \frac{F_{p-(\xi)}}{p} \pmod{p}.
\]

So (1.15) also holds.

Applying (1.11) with \(m = -5\) we get
\[
\frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k5^{k-1}} \equiv V_p(-5) + 5^p \pmod{p}.
\]

As the two roots of the equation \(x^2 + 5x + 5 = 0\) are \((-5 \pm \sqrt{5})/2\), we have
\[
V_p(-5) = \left(\frac{-5 + \sqrt{5}}{2}\right)^p + \left(\frac{-5 - \sqrt{5}}{2}\right)^p
= \sqrt{5}^p \left(\frac{1 - \sqrt{5}}{2}\right)^p - \sqrt{5}^p \left(\frac{1 + \sqrt{5}}{2}\right)^p
= -\sqrt{5}^{p+1}F_p.
\]

Recall that
\[
\left(\frac{5}{p}\right)F_p = L_p - 2F_{p-(\xi)} \equiv 1 + \frac{1}{2}F_{p-(\xi)} \pmod{p^2}.
\]

Thus
\[
5^{(p-1)/2}F_p - 1 \equiv 5^{(p-1)/2} \left(\frac{5}{p}\right) \left(1 + \frac{1}{2}F_{p-(\xi)}\right) - 1
\equiv \left(\frac{5}{p}\right) \left(5^{(p-1)/2} - \left(\frac{5}{p}\right)\right) + \frac{1}{2}F_{p-(\xi)}
\equiv \frac{1}{2} \left(5^{(p-1)/2} + \left(\frac{5}{p}\right)\right) \left(5^{(p-1)/2} - \left(\frac{5}{p}\right)\right) + \frac{1}{2}F_{p-(\xi)}
\equiv 5^{p-1} - 1 + \frac{1}{2}F_{p-(\xi)} \pmod{p^2}
\]

and hence
\[
\frac{1}{2} \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k5^{k-1}} \equiv \frac{5^p - 5(p+1)/2F_p}{p} = \frac{5^p - 5}{p} - 5 \frac{5(p-1)/2F_p - 1}{p}
\equiv 5 \left(q_p(5) - \frac{q_p(5)}{2} - \frac{F_{p-(\xi)}}{2p}\right) \pmod{p}.
\]
This proves (1.16).

(ii) As $2 + 2\sqrt{2}$ and $2 - 2\sqrt{2}$ are the two roots of the equation $x^2 - 4x - 4 = 0$, we have

$$V_p(4) = (2 + 2\sqrt{2})^p + (2 - 2\sqrt{2})^p = 2^p \left( (1 + \sqrt{2})^p + (1 - \sqrt{2})^p \right) = 2^p Q_p,$$

where the sequence $\{Q_n\}_{n \in \mathbb{N}}$ is given by

$$Q_0 = Q_1 = 2 \text{ and } Q_{n+1} = 2Q_n + Q_{n-1} \,(n = 1, 2, 3, \ldots).$$

By [15, Remark 3.1],

$$4 \left( \frac{2}{p} \right) P_p - Q_p = \left( \frac{2}{p} \right) Q_{p-(\frac{p}{2})} \equiv 2 \pmod{p^2}$$

and

$$P_{p-(\frac{p}{2})} = \left( \frac{2}{p} \right) P_p - 1 \pmod{p^2}.$$

Thus

$$Q_p - 2 \equiv 4 \left( \frac{2}{p} \right) P_p - 1 \equiv 4P_{p-(\frac{p}{2})} \pmod{p^2}$$

and hence

$$\sum_{k=1}^{p-1} (-1)^k \frac{2^k}{k4^k} \equiv 4^p - V_p(4) \frac{2^p - Q_p}{2p} \equiv 2^{p-1} 2^p - Q_p \equiv 2q_p(2) - 2 \frac{P_{p-(\frac{p}{2})}}{p} \pmod{p}$$

with the help of (1.11) in the case $m = 4$.

By [14],

$$-2^{(p+1)/2} \frac{P_p - 2^{(p-1)/2}}{p} \equiv \sum_{k=1}^{\frac{p-1}{2}} \frac{1}{k2^k} \equiv \sum_{k=1}^{\lfloor \frac{3p}{4} \rfloor} (-1)^{k-1} \frac{1}{k} \pmod{p}.$$  

(The last congruence was first conjectured by Z. H. Sun in 1988.) Observe that

$$-2^{(p+1)/2} \frac{P_p - 2^{(p-1)/2}}{p} \equiv -2^{(p+1)/2} \frac{2^{(p-1)/2}}{p} (1 + P_{p-(\frac{p}{2})}) - 2^{(p-1)/2}$$

$$\equiv -2 \frac{P_{p-(\frac{p}{2})}}{p} + 2^{(p+1)/2} \frac{2^{(p-1)/2} - (\frac{p}{2})}{p}$$

$$\equiv -2 \frac{P_{p-(\frac{p}{2})}}{p} + q_p(2) \pmod{p}.$$
So we also have
\[ 2q_p(2) - 4 \frac{p^{p-(\frac{2}{3})}}{p} \equiv 2 \sum_{k=1}^{\lfloor \frac{3p}{4} \rfloor} \frac{(-1)^{k-1}}{k} \pmod{p}. \]

(iii) Now suppose \( p > 3 \). By Theorem 1.2 in the case \( m = 2 \), we have
\[ \sum_{k=1}^{p-1} (-1)^k \frac{2k}{k^2} \equiv 2p - V_p(2) \pmod{p}. \]
Observe that the two roots of the equation \( x^2 - 2x - 2 = 0 \) are \( 1 \pm \sqrt{3} \).
Thus
\[ V_p(2) = (1 + \sqrt{3})^p + (1 - \sqrt{3})^p = 2 \sum_{k=0}^{(p-1)/2} \binom{p}{2k} (\sqrt{3})^{2k} \]
\[ = 2 + \sum_{k=1}^{(p-1)/2} 2p \frac{p-1}{2k} \binom{p-1}{2k-1} 3^k \]
\[ \equiv 2 - p \sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \pmod{p^2}. \]
As observed by Eisenstein [2],
\[ 2q_p(2) = 2p - 2 \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \pmod{p}. \]
By a congruence of Z. W. Sun [15],
\[ \sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv \sum_{0 < k < p/6} (-1)^k \frac{p}{k} \equiv \sum_{0 < k < p/6} (-1)^{p-k} \frac{p-k}{k} \pmod{p}. \]
Thus
\[ \sum_{k=1}^{p-1} (-1)^k \frac{2k}{k^2} \equiv 2p - 2 \frac{V_p(2) - 2}{p} \equiv 2q_p(2) + \sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \]
\[ \equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} + \sum_{5p/6 < k < p} (-1)^k \frac{1}{k} \equiv \sum_{0 < k < 5p/6} (-1)^{k-1} \frac{1}{k} \pmod{p}. \]
In light of [15],

\[
\sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv -q_p(2) - 6 \left( \frac{2}{p} \right) \frac{S_{(p-(\frac{3}{2})^2)}}{p} \pmod{p}.
\]

So we also have

\[
\sum_{k=1}^{p-1} (-1)^k \frac{2^k}{k2^k} \equiv q_p(2) - 6 \left( \frac{2}{p} \right) \frac{S_{(p-(\frac{3}{2})^2)}}{p} \pmod{p}.
\]

Therefore (1.18) follows.

Let \( u_n = u_n(2, -2) \) and \( v_n = v_n(2, -2) \) for \( n \in \mathbb{N} \). By induction,

\[
v_n = 2u_{n+1} - 2u_n = 2u_n + 4u_{n-1} \text{ for } n = 1, 2, 3, \ldots.
\]

Thus

\[
v_p = 2 \left( \frac{3}{p} \right) u_p + \left( 3 + \frac{3}{p} \right) u_{p-(\frac{3}{2})}.
\]

Clearly

\[
2\sqrt{3}u_{p-(\frac{3}{2})} = (1 + \sqrt{3})^{p-(\frac{3}{2})} - (1 - \sqrt{3})^{p-(\frac{3}{2})}
\]

\[
= 2^{p-(\frac{3}{2})/2} \left( (2 + \sqrt{3})^{(p-(\frac{3}{2})/2} - (2 - \sqrt{3})^{(p-(\frac{3}{2})/2} \right)
\]

and hence

\[
u_{p-(\frac{3}{2})} = 2^{(p-(\frac{3}{2})/2} S_{(p-(\frac{3}{2})/2)}/2 \equiv \left( \frac{2}{p} \right) 2^{(1-(\frac{3}{2})/2} S_{(p-(\frac{3}{2})/2)}/2 \pmod{p^2}.
\]

Recall that

\[
v_p = V_p(2) = 2 - p \sum_{k=1}^{(p-1)/2} \frac{3^k}{k} \equiv 2 + (2^{p-1} - 1) + 6 \left( \frac{2}{p} \right) S_{(p-(\frac{3}{2})/2)}/2 \pmod{p^2}.
\]

Therefore

\[
2 \left( \frac{3}{p} \right) u_p - 2 = v_p - 2 - \left( 3 + \frac{3}{p} \right) u_{p-(\frac{3}{2})}
\]

\[
\equiv 2^{p-1} - 1 + 6 \left( \frac{2}{p} \right) S_{(p-(\frac{3}{2})/2} - \left( 3 + \frac{3}{p} \right) \left( \frac{2}{p} \right) 2^{(1-(\frac{3}{2})/2} S_{(p-(\frac{3}{2})/2} \pmod{p^2}\]

\[
\equiv 2^{p-1} - 1 + 2 \left( \frac{2}{p} \right) S_{(p-(\frac{3}{2})/2} \pmod{p^2}.
\]
Applying (1.11) with $m = -6$, we get
\[
\frac{1}{2} \sum_{k=1}^{p-1} \binom{2k}{k} \equiv \frac{V_p(-6) + 6^p}{p} \equiv \frac{V_p(-6) + 6}{p} + 6(q_p(2) + q_p(3)) \quad (\text{mod } p).
\]

Observe that
\[
V_p(-6) = (-3 + \sqrt{3})^p + (-3 - \sqrt{3})^p
= -\sqrt{3}^p \left( (1 + \sqrt{3})^p - (1 - \sqrt{3})^p \right) = -2 \times 3^{(p+1)/2} u_p
\]
and hence
\[
V_p(-6) + 6 \equiv -6 \left( 3^{(p-1)/2} - \left( \frac{3}{p} \right) \right) u_p - 6 \left( \frac{3}{p} \right) u_p + 6
\equiv -3 \left( 2^{p-1} - 1 \right) + 2 \left( \frac{2}{p} \right) S_{(p-(\frac{p}{2}))} \quad (\text{mod } p^2).
\]

Therefore
\[
\frac{1}{2} \sum_{k=1}^{p-1} \binom{2k}{k} \equiv -3 \left( q_p(3) + q_p(2) + 2 \left( \frac{2}{p} \right) \frac{S_{(p-(\frac{p}{2}))}}{p} \right) + 6(q_p(2) + q_p(3)) \equiv 3 \left( q_p(2) + q_p(3) - 2 \left( \frac{2}{p} \right) \frac{S_{(p-(\frac{p}{2}))}}{p} \right) \quad (\text{mod } p).
\]

So (1.19) is valid.

The proof of Corollary 1.3 is now complete. □

5. Proof of Theorem 1.3

Proof of Theorem 1.3. By an identity of T. B. Staver [12],
\[
\sum_{k=1}^{n} \binom{2k}{k} = \frac{2n+1}{3n^2} \binom{2n}{n} \sum_{k=1}^{n} \frac{1}{k(k-1)^2} = \frac{n+1}{3} \binom{2n+1}{n} \sum_{k=1}^{n} \frac{1}{k^2(k-1)^2}
\]
for all $n = 1, 2, 3, \ldots$. Taking $n = p^a - 1$ in the identity, we get
\[
\sum_{k=1}^{p^a-1} \frac{1}{k} \binom{2k}{k} = \frac{p^a}{3} \left( 2p^a - 1 \right) \sum_{k=1}^{p^a-1} \frac{1}{k^2} \left( p^a - 1 \right). \tag{5.1}
\]
Recall that 
\[
\left(\frac{2p^a - 1}{p^a - 1}\right) \equiv 1 + p[p = 2] + p^2[p = 3] \pmod{p^3}
\]
by [16, Lemma 2.2]. For \(k = 1, \ldots, p^a - 1\), we set \(H_k = \sum_{0 < j \leq k} 1/j\) and note that
\[
\frac{1}{(p^a - 1)^2} = \prod_{0 < j \leq k} \frac{1}{(1 - p^a/j)^2} = \prod_{0 < j \leq k} \left(1 + \frac{p^a}{j} + \frac{p^{2a}}{j^2}\right)^2
\]
\[
= \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j} + 2\frac{p^{2a}}{j^2}\right) \pmod{p^3}
\]
\[
= \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) \pmod{p^2[p = 3]}.
\]
Therefore (5.1) implies that
\[
p^{a-1} \sum_{k=1}^{p^a-1} \frac{(2k)}{k} = \frac{p}{3} \left(\frac{2p^a - 1}{p^a - 1}\right) \sum_{k=1}^{p^a-1} \frac{p^{2(a-1)} k^2}{p^a k^2 - 1}
\]
\[
\equiv \frac{p}{3} \left(1 + p[p = 2] + p^2[p = 3]\right) \sum_{k=1}^{p^a-1} \frac{p^{2(a-1)} k^2}{p^a k^2} \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) \pmod{p^3}.
\]
So we have
\[
p^{a-1} \sum_{k=1}^{p^a-1} \frac{(2k)}{k} \equiv \left(\frac{p}{3} + p^2[p = 3]\right) \sum_{k=1}^{p^a-1} \frac{p^{2(a-1)} k^2}{p^a k^2} \prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) \pmod{p^3}.
\]
(5.2)

For \(k = 1, \ldots, p^a - 1\), clearly
\[
\prod_{0 < j \leq k} \left(1 + 2\frac{p^a}{j}\right) \equiv 1 + 2p^a H_k + 2p^{2a} \sum_{0 < j < k \leq k} \frac{1}{ij}
\]
\[
\equiv 1 + 2p^a H_k + 2p^{2a} \left(H_k^2 - \sum_{j=1}^{k} \frac{1}{j^2}\right) \pmod{p^3}.
\]
In the case \(a \geq 2\), if \(1 \leq k \leq p^a - 1\) and \(p^a - 2 \nmid k\) then \(p^{2(a-1)/k^2} \equiv
0 \pmod{p^3}). \quad \text{When } a \geq 2 \text{ and } k \in \{1, \ldots, p^2 - 1\}, \text{ we have}

\prod_{j=1}^{p^a - 2k} \left(1 + \frac{2p^a}{j}\right) \equiv 1 + 2 \sum_{j=1}^{p^a} \frac{p^a}{j} + 2 \left(\sum_{j=1}^{p^a} \frac{p^a}{j}\right)^2 - 2 \sum_{j=1}^{p^a} \frac{p^a}{j^2}

\equiv 1 + 2 \sum_{i=1}^{k} \frac{p^a}{p^a - 2i} + 2 \left(\sum_{i=1}^{k} \frac{p^a}{p^a - 2i}\right)^2 - 2 \sum_{i=1}^{k} \frac{p^a}{(p^a - 2i)^2}

\equiv 1 + 2p^2H_k + 2(p^2H_k)^2 - 2 \sum_{i=1}^{k} \frac{p^a}{j^2} \pmod{p^3}.

Therefore, if \( a \geq 2 \) then (5.2) implies that

\( p^{a-1} \sum_{k=1}^{p^{a-1}} \frac{(2k)}{k} \equiv \left(\frac{p}{3} + p^2[p \leq 3]\right) \sum_{k=1}^{p^{a-1}} \frac{p^2(a-1)}{(p^{a-2}k)^2} \prod_{j=1}^{p^{a-1}} \left(1 + \frac{2p^a}{j}\right) \)

\equiv \left(\frac{p}{3} + p^2[p \leq 3]\right) \sum_{k=1}^{p^{a-1}} \frac{p^a}{k^2} \prod_{j=1}^{p^{a-1}} \left(1 + \frac{2p^2}{j}\right) \pmod{p^3}.

In the case \( p = 3 \), this yields (1.20) for \( a \geq 2 \). (1.20) in the case \( p = 3 \) and \( a = 1 \) can be verified directly.

Below we assume that \( p \not= 3 \). For \( k = 1, \ldots, p^a - 1 \), if \( p^a - 1 \mid k \) then \( p^{2(a-1)}/k^2 \equiv 0 \pmod{p^2} \). Also,

\( p^aH_{p^{a-1}k} = \sum_{j=1}^{p^{a-1}} \frac{p^a}{j} \equiv \sum_{i=1}^{k} \frac{p^a}{p^{a-1}i} = pH_k \pmod{p^2} \)

for every \( k = 1, \ldots, p - 1 \). Thus (5.2) implies that

\( p^{a-1} \sum_{k=1}^{p^{a-1}} \frac{(2k)}{k} \equiv \left(\frac{p}{3} + p^2[p = 2]\right) \sum_{k=1}^{p^{a-1}} \frac{p^2(a-1)}{(p^{a-1}k)^2} (1 + 2p^aH_{p^{a-1}k}) \)

\equiv \left(\frac{p}{3} + p^2[p = 2]\right) \sum_{k=1}^{p^{a-1}} \frac{1 + 2pH_k}{k^2} \pmod{p^3}.

This yields (1.20) in the case \( p = 2 \).

Now we handle the remaining case \( p > 3 \). By the above, it suffices to show that

\[ \sum_{k=1}^{p^{a-1}} \frac{1 + 2pH_k}{k^2} \equiv \frac{8}{3} pB_{p-3} \pmod{p^2}. \quad (5.3) \]
Let \( n \in \mathbb{N} \). It is well known that
\[
\sum_{j=0}^{k-1} j^n = \frac{1}{n+1} \sum_{i=0}^{n} \binom{n+1}{i} B_i k^{n+1-i} \quad \text{for } k \in \mathbb{Z}^+,
\]
and that
\[
\sum_{k=1}^{p-1} k^n \equiv pB_n \equiv 0 \pmod{p} \quad \text{if } n \not\equiv 0 \pmod{p-1}.
\]
(See, e.g., [6, p. 235].) Therefore
\[
\sum_{k=1}^{p-1} \sum_{j=0}^{k-1} j^p \equiv \sum_{k=1}^{p-1} \left( k^p - \frac{1}{k^2(p-1)} \sum_{i=0}^{p-2} \binom{p}{i} B_i k^{p-1-i} \right)
\]
\[
= \sum_{k=1}^{p-1} k^p - \frac{1}{p-1} \sum_{i=0}^{p-2} \binom{p}{i} B_i \sum_{k=1}^{p-1} k^{p-3-i}
\]
\[
\equiv \frac{(p-1)}{p-3} B_{p-3} + \frac{B_{p-2}}{2} \sum_{k=1}^{p-1} \left( \frac{1}{k} + \frac{1}{p-k} \right) \pmod{p}
\]
and hence
\[
\sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv B_{p-3} \pmod{p}. \quad (5.4)
\]
By a result of J. W. L. Glaisher [3, 4],
\[
\binom{2p-1}{p-1} \equiv 1 - p^2 \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 1 - \frac{2}{3} p^3 B_{p-3} \pmod{p^4}
\]
and thus
\[
\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv \frac{2}{3} pB_{p-3} \pmod{p^2}. \quad (5.5)
\]
Note that (5.3) follows from (5.4) and (5.5). We are done. \( \square \)

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